Remark about projections:

There exists a bijective relation between projections and subspaces of $\mathcal{H}$.
It means $\mathrm{Ran}(P)$ is a subspace of $\mathcal{H}$,
and there exists a unique projection on any subspace.
We write $P = P_M$ ← underlying subspace

⚠ $P_M P_N \neq P_N P_M$ in general $\Rightarrow$ $P_M P_N$ is not a projection in general.

Def. $U \in \mathcal{B}(\mathcal{H})$ is unitary if $UU^* = 1$ and $U^*U = 1$.

⚠ When $\dim \mathcal{H} = \infty$, these 2 conditions are not equivalent.
Note that for U unitary, $U^{-1} = U^*$ (in particular $U^{-1} \in \mathcal{B}(\mathcal{H})$)

Def. $V \in \mathcal{B}(\mathcal{H})$ is an isometry if $V^*V = 1$.

Remark: for $f \in \mathcal{H}$

$$\|Vf\|^2 = \langle Vf, Vf\rangle = \langle V^*Vf, f\rangle = \langle f, f\rangle = \|f\|^2$$

$$\Rightarrow \|Vf\| = \|f\| \quad \text{(isometry)}$$

Also $\langle Vf, Vg\rangle = \langle f, g\rangle$

Question: give an example (e.g. in $\ell^2(\mathbb{N})$)

Remark: $P := VV^*$ is a projection on $\mathrm{Ran}(V)$. Indeed

- $P^2 = (VV^*)(VV^*) = V(V^*V)V^* = VV^* = P$;
  $P^* = (VV^*)^* = (V^*)^*V^* = VV^* = P$.
- $\mathrm{Ran}(P) = \mathrm{Ran}(VV^*) \subset \mathrm{Ran}(V)$;
  $\forall g \in \mathrm{Ran}(V): \exists f \in \mathcal{H}: g = Vf$
  $\Rightarrow Pg = VV^*g = VV^*Vf = Vf = g$

  Thus P acts as identity on $\mathrm{Ran}(V)$, which means $\mathrm{Ran}(P) = \mathrm{Ran}(V)$.

$\Rightarrow P$ is a projection on $\mathrm{Ran}(V)$.

V is injective since $Vf = 0 \Rightarrow f = 0$.

Hence V is invertible. But its inverse is not always bounded.

More generally:

Def. $W \in \mathcal{B}(\mathcal{H})$ is a partial isometry if $W^*W =: Q$ is a projection.

Lemmas: (as exercises)

- $WQ = W$
- $PW = W$
- $WW^* =: P$

Reference: [Amr] Prop. 2.1

Q is called the initial set projection, and P the final set projection.

Let $\{g_j, h_j\}_{j=1}^N \subset \mathcal{H}$, and for any $f \in \mathcal{H}$ we define

$$Af := \sum_{j=1}^N |h_j\rangle\langle g_j| f \equiv \sum_{j=1}^N \underbrace{\langle g_j, f\rangle}_{=0 \text{ if } f\in\{g_j\}_j^\perp} h_j \in \text{Vect}\{h_j\}_j$$

Af is clearly linear in f; and

$$\|Af\| \le \sum_{j=1}^N \|\underbrace{\langle g_j, f\rangle}_{\in\mathbb{C}} h_j\| = \sum_{j=1}^N |\langle g_j, f\rangle| \|h_j\| \le \sum_{j=1}^N \|f\|\|g_j\|\|h_j\|$$

$$\Rightarrow \|A\| = \sup_{f\in\mathcal{H}; f\ne 0} \frac{\|Af\|}{\|f\|} \le \sum_{j=1}^N \|g_j\|\|h_j\|$$

$\Rightarrow \|A\| \in \mathcal{B}(\mathcal{H})$

Such A is called a finite rank operator. Its set is denoted as $\mathcal{F}(\mathcal{H})$.

Remark:

- $\underset{\in\mathcal{H}}{h_j} \otimes \underset{\in\mathcal{H}^*}{g_j} := |h_j\rangle\langle g_j|$.
- $Af = 0$ if $f \notin \text{Vect}\{g_j\}$; $\text{Ran}(Af) \subset \text{Vect}\{h_j\}$. Both are finite dimensional

Def. An element $K \in \mathcal{B}(\mathcal{H})$ is compact if

$\forall \varepsilon > 0 : \exists A$ finite rank operator s.t. $\|K - A\| \le \varepsilon$.

We write $\mathcal{K}(\mathcal{H})$ for the set of all compact operators. subset of $\mathcal{B}(\mathcal{H})$

Remark:

$\overline{\mathcal{F}(\mathcal{H})}^{u} = \mathcal{K}(\mathcal{H})$ in the norm topology; (closure)

$\overline{\mathcal{K}(\mathcal{H})}^{u} = \mathcal{K}(\mathcal{H})$ in the norm topology;

$\overline{\mathcal{K}(\mathcal{H})}^{s} = \mathcal{B}(\mathcal{H}) = \overline{\mathcal{K}(\mathcal{H})}^{w}$ strong or weak topologies.

Example: Consider $\mathcal{H} = L^2(\mathbb{R})$, and

$\varphi_n : \mathbb{R} \to \mathbb{R}$, $\varphi_n(x) = \begin{cases} 1 & \text{if } x \in [n, n+1] \\ 0 & \text{otherwise} \end{cases}$ with $n \in \mathbb{N}$.

Consider $B_n = M_{\varphi_n}$. Then

$\text{s-}\lim_{n\to 0} B_n = \mathbf{0}$ but $\text{u-}\lim_{n\to 0} B_n$ does not exist.

For s-lim, consider a fixed $f \in \mathcal{H}$ and observe that

$$\|(B_n - \mathbf{0})f\|^2 = \int |[B_n f](x)|^2 dx = \int_n^{n+1} |f(x)|^2 dx \xrightarrow[n\to\infty]{\text{by contradiction in } L^2(\mathbb{R})} 0$$

For u-lim, we cannot fix f, and

$\forall n \in \mathbb{N} : \exists f_n \in \mathcal{H}, \|f_n\| = 1, \text{supp} f_n \subset [n, n+1]$:

$$\|(B_n - \mathbf{0})f_n\|^2 = \int |[B_n f_n](x)|^2 dx = \int_n^{n+1} |f_n(x)|^2 dx = 1$$

which implies that $\text{u-}\lim_{n\to 0} B_n$ does not exist.

$\mathcal{K}(\mathcal{H})$ is already a very rich set:

non-commutative integration theory, trace class op.,

Hilbert-Schmidt, Schatten classes are under $\mathcal{K}(\mathcal{H})$.

Prop. Let $K \in \mathcal{K}(\mathcal{H})$, $\{f_n\}_n \subset \mathcal{H}$, $\{B_n\}_n \subset \mathcal{B}(\mathcal{H})$.

1) $\text{w-}\lim_{n\to\infty} f_n = f \in \mathcal{H} \Rightarrow \text{s-}\lim_{n\to\infty} Kf_n = Kf$

2) $\text{s-}\lim_{n\to\infty} B_n = B \in \mathcal{B}(\mathcal{H}) \Rightarrow \text{u-}\lim_{n\to\infty} B_n K = BK$ and $\text{u-}\lim_{n\to\infty} KB_n^* = KB^*$.

I.4 Vector valued and operator valued functions

Consider $[a,b] \subset \mathbb{R}$, and

$\varphi : [a,b] \longmapsto \mathcal{H}$, $\varphi(x) \in \mathcal{H}$ — a vector valued function

or :

$\psi : [a,b] \longmapsto \mathcal{B}(\mathcal{H})$, $\varphi(x) \in \mathcal{B}(\mathcal{H})$ — an operator valued function

Continuity or differentiability can be associated to the different notions of convergence (= different topology).

Example:

φ is strongly continuous at x if

$$\|\varphi(x+\varepsilon) - \varphi(x)\| \xrightarrow{\varepsilon \to 0} 0$$

or ψ is uniformly differenciable at x if $\exists\, \psi'(x) \in \mathcal{B}(\mathcal{H})$ with

$$\left\| \frac{\psi(x+\varepsilon) - \psi(x)}{\varepsilon} - \psi'(x) \right\| \xrightarrow{\varepsilon \to 0} 0$$

Also, Riemann's type integral exists.

$$\int_a^b \varphi(x)\,dx := \underset{\substack{\text{finite partitions} \\ \text{of } [a,b]}}{\text{s-lim}} \sum_{j=1}^{N} (x_k - x_{k-1})\, \varphi(y_k) \in \mathcal{H} \text{ if the limit exists.}$$

$x_{k-1} < x_k$; $x_{k-1} \leq y_k \leq x_k$

Then

$\varphi \longmapsto \int_a^b \varphi(x)\,dx \in \mathcal{H}$ is a linear map ;

and one has

$$\left\| \int_a^b \varphi(x)\,dx \right\| \leq \int_a^b \|\varphi(x)\|\,dx$$

Similarly one can define

$$\int_a^b \psi(x)\,dx \in \mathcal{B}(\mathcal{H})$$

Question:

If $\psi(x) \in \mathcal{K}(\mathcal{H})$ $\forall x \in [a,b]$, when does $\int_a^b \psi(x)\,dx \in \mathcal{K}(\mathcal{H})$?

Depends on the property of

$[a,b] \ni x \longmapsto \psi(x) \in \mathcal{K}(\mathcal{H})$