Def. $\|\varphi\|_{\mathcal{H}^*} = \sup\limits_{\substack{f \in \mathcal{H}^* \\ f \neq 0}} \frac{|\varphi(f)|}{\|f\|} < \infty$

Lemma: (Rietz Lemma)

$\forall \varphi \in \mathcal{H}^* \exists ! g \in \mathcal{H} : \varphi(f) = \langle g, f \rangle \ \forall f \in \mathcal{H}.$

$\leadsto$ identification of $\mathcal{H}$ with $\mathcal{H}^*$ anti-linear

I.2 Bounded linear operators

If M, N are vector spaces, $B : M \longmapsto N$ is a linear operator (= map) if

$B(\alpha f + g) = \alpha B(f) + B(g)$

Def. A linear map $B : \mathcal{H} \mapsto \mathcal{H}$ is bounded if

$\exists c > 0 : \|B(f)\| \leq c \|f\| \ \forall f \in \mathcal{H}$

We set $\mathcal{B}(\mathcal{H})$ [$\equiv \mathcal{L}(\mathcal{H})$ sometimes in old books] for the set of all bounded linear maps on $\mathcal{H}$

The norm of $B \in \mathcal{B}(\mathcal{H})$ is

$\|B\| := \sup\limits_{\substack{f \in \mathcal{H} \\ f \neq 0}} \frac{\|Bf\|}{\|f\|} = \sup\limits_{\|f\| = 1} \|Bf\|$

Remark:

- Range of B is

  $\mathrm{Ran}(B) := \{ g \in \mathcal{H} \mid g = Bf \ \exists f \in \mathcal{H} \} =: B\mathcal{H}$

- $\|B\| = \sup\limits_{\substack{f \in M_1, g \in M_2 \\ \|f\| = \|g\| = 1}} |\langle f, Bg \rangle|$ for M_1, M_2 dense linear manifolds.

  If $\mathcal{H} = L^2(\mathbb{R})$, $M_1 = \mathcal{S}(\mathbb{R})$, $C_c^\infty(\mathbb{R})$ or $C_c(\mathbb{R})$
  with compact support

Lemma: For any $B \in \mathcal{B}(\mathcal{H})$ $\exists ! B^* \in \mathcal{B}(\mathcal{H})$ called the adjoint satisfying

$\langle B^* f, g \rangle = \langle f, Bg \rangle.$

In addition,

$\|B^*\| = \|B\|$; $(B^*)^* = B$; $\|B^* B\| \overset{\uparrow}{=} \|B\|^2$; $(AB)^* = B^* A^*$.

C*-property $\leadsto$ C*-algebra

Hint: Consider for fixed $f \in \mathcal{H}$,

$g \xmapsto[\text{linear}]{\text{bounded}} \langle f, Bg \rangle \xrightarrow[\text{lemma}]{\text{Rietz}} \langle f^*, g \rangle$

Now we define

$B^* : \mathcal{H} \mapsto \mathcal{H}, \ f \longmapsto f^*$

$\Rightarrow \mathcal{B}(\mathcal{H})$ is a C^*-algebra.

Remark: For $\mathcal{H} = \mathbb{C}^n$, and $B = (b_{ij}) \in M^n(\mathbb{C})$, then $B^* = (\overline{b_{ji}})$

Counterexample: $\cancel{\mathcal{B}(\mathbb{C}^n) =}$ $M^n(\mathbb{C})$ with the norm $|||a_{ij}||| = \sup\limits_{i,j} |a_{ij}|$ is not a C^*-algebra.

About convergences:

Def. Let $\{B_n\} \in \mathcal{B}(\mathcal{H})$.

$$\underset{n\to\infty}{u\text{-}\lim} B_n = B_\infty \in \mathcal{B}(\mathcal{H}) \text{ iff } \lim_{n\to\infty} \|B_n - B_\infty\| = 0 \quad \text{(uniform)}$$

$$\Downarrow \not\Uparrow$$

$$\underset{n\to\infty}{s\text{-}\lim} B_n = B_\infty \in \mathcal{B}(\mathcal{H}) \text{ iff } \lim_{n\to\infty} \|(B_n - B_\infty) f\| = 0 \quad \forall f \in \mathcal{H} \quad \text{(strong)}$$

$$\Downarrow \not\Uparrow$$

$$\underset{n\to\infty}{w\text{-}\lim} B_n = B_\infty \in \mathcal{B}(\mathcal{H}) \text{ iff } \lim_{n\to\infty} \langle g, (B_n - B_\infty) f\rangle = 0 \quad \forall f, g \in \mathcal{H} \quad \text{(weak)}$$

Exercise: examples

Remark:

- $\underset{n\to\infty}{w\text{-}\lim} B_n = B_\infty \Rightarrow \underset{n\to\infty}{w\text{-}\lim} B_n^* = B_\infty^*$
- $\underset{n\to\infty}{s\text{-}\lim} A_n = A_\infty$ and $\underset{n\to\infty}{s\text{-}\lim} B_n = B_\infty \Rightarrow \underset{n\to\infty}{s\text{-}\lim} A_n B_n = A_\infty B_\infty$ tricky to prove; based on uniform boundedness principle.

Def. (Invertibility)

$B \in \mathcal{B}(\mathcal{H})$ is invertible if $Bf = 0 \Rightarrow f = 0$ (injectivity)

Indeed $Bf = Bg \Leftrightarrow B(f-g) = 0 \Rightarrow f - g = 0 \Leftrightarrow f = g$

Then we set

$B^{-1} : \operatorname{Ran}(B) \longmapsto \mathcal{H}$,

satisfying

$B^{-1}Bf = f \quad \forall f \in \mathcal{H}; \quad BB^{-1}g = g \quad \forall g \in \operatorname{Ran}(B)$.

B is boundedly invertible if

$\|B^{-1}g\| \leq c\|g\| \quad \forall g \in \operatorname{Ran}(B)$

In particular,

$\operatorname{Ran}(B) = \mathcal{H} \Rightarrow B^{-1} \in \mathcal{B}(\mathcal{H}) \Leftrightarrow B$ is boundedly invertible based on closed graph thm.

⚠ If $\operatorname{Ran}(B) \neq \mathcal{H}$, it is possible that B^{-1} is an unbounded operator.

Example:

Consider $\mathcal{H} \in L^2(\mathbb{R})$, and for $\varphi : \mathbb{R} \mapsto \mathbb{C}$, bounded (more precisely $\varphi \in L^\infty(\mathbb{R})$)

set $M_\varphi \equiv \varphi(X) \equiv \varphi(Q)$ multiplication operator $\in \mathcal{B}(\mathcal{H})$ with

$[M_\varphi f](x) := \varphi(x) f(x) \quad \forall f \in \mathcal{H},\ x \in \mathbb{R}$.

Clearly M_φ is linear and

$$\|M_\varphi f\|_{L^2}^2 = \int_{\mathbb{R}} |[M_\varphi f](x)|^2 dx = \int_{\mathbb{R}} |\varphi(x) f(x)|^2 dx = \int_{\mathbb{R}} |\varphi(x)|^2 |f(x)|^2 dx$$
$$\leq \operatorname*{ess\,sup}_{x\in\mathbb{R}} |\varphi(x)|^2 \int_{\mathbb{R}} |f(x)|^2 dx = \|\varphi\|_\infty^2 \|f\|_{L^2}^2$$

$\Rightarrow M_\varphi \in \mathcal{B}(\mathcal{H}); \quad \|M_\varphi\| = \|\varphi\|_\infty$

One has $M_\varphi^* = M_{\overline{\varphi}}$.

Indeed $\langle f, M_\varphi g\rangle = \int \overline{f(x)\,\overline{\overline{\varphi(x)}}}\, g(x)\,dx = \int \overline{f(x)\,\overline{\varphi(x)}}\, g(x)\,dx = \langle M_{\overline{\varphi}} f, g\rangle$

Example:

Consider $\varphi(x) := \tanh(x)$. Then $M_{\tanh} \in \mathcal{B}(\mathcal{H})$.

One has $M_{\tanh}^{-1} = M_{\frac{1}{\tanh}}$ because

$M_{\tanh} M_{\frac{1}{\tanh}} = M_{\frac{1}{\tanh}} M_{\tanh} = 1$.

But $\frac{1}{\tanh}$ is not bounded. $\Rightarrow$ $M_{\frac{1}{\tanh}} \notin \mathcal{B}(\mathcal{H})$

One says that $M_{\tanh}$ is not invertible in $\mathcal{B}(\mathcal{H})$ or not boundedly invertible.

Remark:

$\mathbf{0} \in \mathcal{B}(\mathcal{H})$, $\mathbf{0} f = 0$;

$1 \in \mathcal{B}(\mathcal{H})$, $1 f = f$

Lemma: (Newmann series)

If $B \in \mathcal{B}(\mathcal{H})$ with $\|B\| < 1$ then

$$(1-B)^{-1} = \sum_{n=0}^{\infty} B^n \text{ with } B^0 = 1 \text{ and } \|(1-B)^{-1}\| \leq (1-\|B\|)^{-1}$$

I.3 Special classes of operators

Def. $B \in \mathcal{B}(\mathcal{H})$ is self-adjoint (or Hermitian) if $B^* = B$.

In the previous example, $M_\varphi^* = M_\varphi$ iff φ is real valued.

Remark:

For self-adjoint operators

$$\|B\| = \sup_{\substack{f \in M \\ f \neq 0}} |\langle f, Bf\rangle| \quad (M \text{ is dense linear manifold})$$

Def. $P \in \mathcal{B}(\mathcal{H})$ is a projection (orthogonal projection) if $P^2 = P = P^*$.

In the previous example,

$M_\varphi^2 = M_\varphi M_\varphi = M_{\varphi^2} \overset{\text{projection}}{=\!=\!=} M_\varphi = M_{\overline{\varphi}} \Leftrightarrow \varphi^2 = \varphi = \overline{\varphi}$

φ is a characteristic function.