Operator Theory on Hilbert spaces

Chapter I : Hilbert spaces and bounded operators

I.1 : Hilbert spaces

Def. A complex Hilbert space ($\nearrow$ denoted by $\mathcal{H}$) is a vector space over $\mathbb{C}$ (stable for addition, and for multiplication by complex numbers) with a scalar product (inner product)

$$\mathcal{H} \times \mathcal{H} \ni (f,g) \longmapsto \langle f,g \rangle \in \mathbb{C}$$

which is linear in the second argument, and anti-linear in the first one, which is complete for the norm defined by $\|f\| := \langle f,f \rangle^{\frac{1}{2}}$ see p. 17
and seperable (existence of a countable basis).

Linearity and anti-linearity above :

$$\langle f, \lambda g + h \rangle = \lambda \langle f,g \rangle + \langle f,h \rangle$$
$$\langle \lambda f + h, g \rangle = \overline{\lambda} \langle f,g \rangle + \langle h,g \rangle$$

The scalar product must satisfy

$$\langle f,g \rangle = \overline{\langle g,f \rangle} \quad \text{and} \quad \langle f,f \rangle \geq 0 \text{ with equality iff } f = 0.$$

Examples

1) $\mathbb{C}$ with $\langle \alpha, \beta \rangle = \overline{\alpha}\beta \ \forall \alpha,\beta \in \mathbb{C}$ 2) $\mathbb{C}^n$ with $\langle \alpha,\beta \rangle = \sum_{j=1}^{n} \overline{\alpha_j} \beta_j \ \forall \alpha,\beta \in \mathbb{C}^n$

3) $\ell^2(\mathbb{Z}) := \{(a_j)_{j\in\mathbb{Z}} \mid a_j \in \mathbb{C}, \sum_{j\in\mathbb{Z}} |a_j|^2 < \infty$ with $\langle (a_j),(b_j) \rangle = \sum_{j\in\mathbb{Z}} \overline{a_j}\, b_j$

4) $L^2(\mathbb{R}^n) := \{[f] : \mathbb{R}^n \longmapsto \mathbb{C}, \text{ measurable}, \int_{\mathbb{R}^n} |f(x)|^2\, dx < \infty\}$
with $\langle f,g \rangle = \int_{\mathbb{R}^n} \overline{f(x)}\, g(x)\, dx$

⚠ For $k \in \mathbb{R}^n$, $\mathbb{R}^n \ni x \longmapsto \exp(-ik\cdot x) \in \mathbb{C}$ (scalar product in $\mathbb{R}^n$) does NOT belong to $L^2(\mathbb{R}^n)$
but if $\varphi \in \mathcal{S}(\mathbb{R}^n)$ (Schwartz space) then $x \longmapsto \int_{\mathbb{R}^n} \exp(-ik\cdot x)\, \varphi(k)\, dk$ belongs to $L^2(\mathbb{R}^n)$
(avarage over some k)

Remarks :

1) All Hilbert spaces are isomorphic either to $\mathbb{C}^n$ for some $n \in \mathbb{N}$, or to $\ell^2(\mathbb{Z})$.

2) One has the polarization identity :

$$4\langle f,g \rangle = \|f+g\|^2 - \|f-g\|^2 - i\|f+ig\|^2 + i\|f-ig\|^2$$ Exercise : check it

3) def $f \perp g$ iff $\langle f,g \rangle = 0$.

Standard inequalities (proof as an exercise)

- $|\langle f,g \rangle| \leq \|f\|\,\|g\|$ Schwarz inequality
- $\|f+g\| \leq \|f\| + \|g\|$ triangle inequality
- $\|f+g\|^2 \leq 2\|f\|^2 + 2\|g\|^2$
- $\big|\|f\| - \|g\|\big| \leq \|f\| - \|g\|$

Def. Let $\{f_n\}_{n\in\mathbb{N}} \subset \mathcal{H}$. We write

$$\underset{n\to\infty}{s\text{-}\lim} f_n = f_\infty (\in \mathcal{H}) \text{ if } \lim_{n\to\infty} \|f_n - f_\infty\| = 0 ;$$ strong limit

$$\underset{n\to\infty}{w\text{-}\lim} f_n = f_\infty (\in \mathcal{H}) \text{ if } \lim_{n\to\infty} \langle g, f_n - f_\infty \rangle = 0 \ \forall g \in \mathcal{H}.$$ weak limit

Example: in $\ell^2(\mathbb{Z}) \ni f_n$,

Consider $f_n(j) = \delta_{jn} = \begin{cases} 1 & \text{if } j = n \\ 0 & \text{otherwise} \end{cases}$ then

$\underset{n\to\infty}{w\text{-}\lim} f_n = \mathbf{0}$; $\underset{n\to\infty}{s\text{-}\lim} f_n$ does not exist. (exercise)

Remark: when these limits exist, they are unique.

(use uniform boundedness principle)

Lemma: for $\{f_n\}_{n\in\mathbb{N}} \subset \mathcal{H}$, one has

$$\underset{n\to\infty}{s\text{-}\lim} f_n = f_\infty \iff \underset{n\to\infty}{w\text{-}\lim} f_n = f_\infty \text{ and } \lim_{n\to\infty} \|f_n\| = \|f_\infty\|$$

Sketch of the proof:

「$\Rightarrow$」 $|\langle g, f_n - f_\infty\rangle| \le \|g\|\|f_n - f_\infty\|$ and $\big|\|f_n\| - \|f_\infty\|\big| \le \|f_n - f_\infty\|$

「$\Leftarrow$」 $\|f_n - f_\infty\|^2 = \langle f_n - f_\infty, f_n - f_\infty\rangle = \|f_n\|^2 + \|f_\infty\|^2 - \langle f_n, f_\infty\rangle - \langle f_\infty, f_n\rangle$

Exercise: complete this proof. … □

Def. A linear manifold is a subset of $\mathcal{H}$ which is stable for addition, and for multiplication by complex numbers. sometimes called a subspace

A subspace is a closed (any Cauchy sequence converges inside) linear manifold.

⇝ A subspace is a Hilbert space with the inherited scalar product.

Examples

1) $\mathcal{S}(\mathbb{R}^n)$ is a linear manifold in $L^2(\mathbb{R}^n)$, but not a subspace.

(the closure of $\mathcal{S}(\mathbb{R}^n)$ in $L^2(\mathbb{R}^n)$ is $L^2(\mathbb{R}^n)$)

2) If $\{f_1, .. f_N\} \subset \mathcal{H}$, then

$\mathrm{Vect}\{f_1, \cdots, f_N\} \equiv \mathrm{Span}\{f_1, \cdots, f_N\} := \left\{\sum_{j=1}^N a_j f_j \,\middle|\, a_j \in \mathbb{C}\right\}$

is a subspace isomorphic to $\mathbb{C}^m$ for some $m \in \{1, \cdots, N\}$

3) If $M \subset \mathcal{H}$ is any subset of $\mathcal{H}$, then

$M^\perp := \{f \in \mathcal{H} \mid \langle f, g\rangle = 0 \ \forall g \in M\}$ is a subspace of $\mathcal{H}$ (exercise)

$M^\perp$ is called the orthocomplement of M.

Def. M is dense in $\mathcal{H}$ iff

$$\forall f \in \mathcal{H} \ \forall \varepsilon > 0 \ \exists g \in M : \|f - g\| \le \varepsilon$$

Lemma: A linear manifold M is dense in $\mathcal{H}$ iff $M^\perp = \{0\}$. (exercise)

Proposition: (Projection Theorem)

Let M be a subspace of $\mathcal{H}$. Then

$$\forall f \in \mathcal{H} \ \exists!\ f_1 \in M, f_2 \in M^\perp : f = f_1 + f_2 \text{ and } \|f\|^2 = \|f_1\|^2 + \|f_2\|^2.$$

Def. (Dual of $\mathcal{H}$)

$$\mathcal{H}^* := \{ \text{bounded linear functional on } \mathcal{H} \}$$
$$\varphi : \mathcal{H} \longmapsto \mathbb{C} \text{ with}$$

$\varphi(f + \lambda g) = \varphi(f) + \lambda \varphi(g)$ (linearity)

$|\varphi(f)| \leq c \|f\|$ for some $c \geq 0$ independent of $f \in \mathcal{H}$ (bounded)

Example :

Take $g \in \mathcal{H}$ and set $\varphi_g(f) := \langle g, f \rangle$

Then one has automatically the linearity on f and

$$|\varphi_g(f)| = |\langle g, f \rangle| \leq \underbrace{\|g\|}_{= c} \|f\| \quad \text{(boundedness)}$$