Def. The set of all Lebesgue integrable functions on $[a,b]$ is is denoted by $\mathcal{L}([a,b])$

⚠ We do not impose boundedness:

$$\int_a^b f_N(x)\,dx \xrightarrow{N\to\infty} \text{converge to } \text{“}\int_a^b f(x)\,dx\text{”}$$

Remark: If $f=g$ a.e. and f is L. integrable the g is L. integrable and $\int f = \int g$.

For integration on unbounded intervals:

If $f \geq 0$, we say that $f \in \mathcal{L}([a,\infty))$ if

$\forall N > a: f|_{[a,N]} \in \mathcal{L}([a,N])$; and $\lim_{N\to\infty} \int_a^N f|_{[a,N]}(x)\,dx$

Then we write

$$\int_a^\infty f(x)\,dx := \lim_{N\to\infty} \int_a^N f(x)\,dx$$

For general $f = f_+ - f_-$ with $f_+, f_- \geq 0$:

$f \in \mathcal{L}([a,\infty))$ iff $f_+, f_- \in \mathcal{L}([a,\infty)) \overset{\text{in } \mathbb{R}}{\Longleftrightarrow} |f| \in \mathcal{L}([a,\infty))$

use this for complex-valued f

Similarly one defines $f \in \mathcal{L}((-\infty,a])$ and $f \in \mathcal{L}(\mathbb{R})$.

Thm. (Lebesgue dominated convergence theorem)

Let $\{f_n\}$ be sequence of L. m. functions on $[a,b]$ s.t.

$\lim_{n\to\infty} f_n(x) = f(x)$ a.e.

Suppose $\exists g \in \mathcal{L}([a,b])$ (dominating function) such that

$\forall n: |f_n| \leq g$ $[\Leftrightarrow |f_n(x)| \leq g(x)\ \forall x \in [a,b]]$ g is indep of n!

Then f is L. integrable and

$$\lim_{n\to\infty} \int_a^b f_n(x)\,dx = \int_a^b \lim_{n\to\infty} f_n(x)\,dx = \int_a^b f(x)\,dx$$

A very useful result!

Example:

For $n \in \mathbb{N}^*$, $x \in [0,1]$, set

$$f_n(x) = \frac{n \sin(x)}{1+n^2\sqrt{x}} + 2e^{x/n}$$

Observe

$f_n(x) \xrightarrow{n\to\infty} 2$ for any x fixed.

And for $x \neq 0$:

$$|f_n(x)| \leq \left|\frac{n}{1+n^2\sqrt{x}}\right| + 2e \leq \frac{1}{n\sqrt{x}} + 2e \leq \frac{1}{\sqrt{x}} + 2e$$

The function $g: [0,1] \mapsto \mathbb{R}$,

$$g(x) = \begin{cases} 2, & x = 0 \\ \frac{1}{\sqrt{x}} + 2e, & 0 < x \leq 1 \end{cases}$$

is L. integrable.

Then by LDCT,

$$\lim_{n\to\infty} \int_0^1 f_n(x)\,dx = \int_0^1 2\,dx = 2$$

Remark: this theory generalizes to measurable functions

$f: I \longmapsto \mathbb{R}$.

box in $\mathbb{R}^n$

But there exists one important thm in n dimensions.

Thm. (Fubini thm) (in $\mathbb{R}^2$) ← report on this

$$\iint f(x,y)\,\underbrace{dx\,dy}_{\text{measure in 2D}} = \int\Big[\int f(x,y)\,\underbrace{dx}\Big]\,\underbrace{dy} = \int\Big[\int f(x,y)\,\underbrace{dy}\Big]\,\underbrace{dx}$$

measures in 1D

⚠ Not true for all f. Some conditions are necessary.

4) L^p - spaces

For $f, g \in \mathcal{L}([a,b])$ we write $f \sim g$ iff $f = g$ a.e.

Exercise: $\sim$ is an equivalence relation.

1) Symmetry: $f \sim f$; 2) Reflexive: $f \sim g \Leftrightarrow g \sim f$;

3) Transitivity: $f \sim g \wedge g \sim h \Rightarrow f \sim h$

Def. $L^1([a,b]) := \mathcal{L}([a,b])/\sim$ (quotion)

We should write $[f] \in L^1([a,b])$ but we simply write $f \in L^1([a,b])$

We set $\|f\|_1 := \int_a^b |f(x)|\,dx$

Exercise: $\|\cdot\|_1$ defines a norm on $L^1([a,b])$.

1) $\|f\|_1 \geq 0$ with equality iff $f = 0$; 2) $\|\lambda f\|_1 = |\lambda|\,\|f\|_1 \;\; \forall \lambda \in \mathbb{R}$;

3) $\|f+g\|_1 \leq \|f\|_1 + \|g\|_1$

Proposition: $L^1([a,b])$ is a Banach space with the norm $\|\cdot\|_1$.

1) L^1 is a vector space with a norm;

2) L^1 is complete [$\Leftrightarrow$ every Cauchy sequence is converging in L^1]

$\{f_n\} \subset L^1$ is a Cauchy sequence iff

$$\forall \varepsilon > 0 \;\exists N \in \mathbb{N} : \forall n, m \geq N : \|f_n - f_m\|_1 \leq \varepsilon$$

Completeness $\Leftrightarrow \forall \{f_n\}$ Cauchy sequence $\exists f_\infty \in L^1 : f_n \xrightarrow[\text{in } L^1\text{-norm}]{n\to\infty} f_\infty$

$\Updownarrow$

$\|f_n - f_\infty\|_1 \xrightarrow{n\to\infty} 0$

Remark: $\mathbb{R}$ and $\mathbb{C}$ are Banach spaces.

⚠ There exists a big difference between pointwise convergence and convergence in L^1-norm.

Example 1:

For $n \in \mathbb{N}^*$, set $f_n : [0,1] \mapsto \mathbb{R}$,

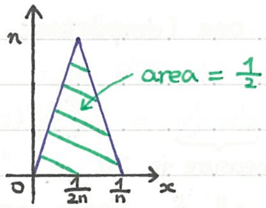
$$f_n(x) = \begin{cases} 0, & x = 0 \\ n, & x \in (0, \frac{1}{n}) \\ \frac{1}{x}, & x \in [\frac{1}{n}, 1] \end{cases}$$

Now $\forall x \in [0,1]$ fixed:

$$f_n(x) \xrightarrow{n\to\infty} \left\{ \begin{matrix} 0, & x = 0 \\ \frac{1}{x}, & x \in (0,1] \end{matrix} \right\} =: f_\infty$$

$$\Rightarrow \underset{L^1([0,1])}{\underset{\cap}{f_n}} \xrightarrow[\text{pointwise}]{n\to\infty} \underset{L^1([0,1])}{\underset{\not\cap}{f_\infty}}$$

Example 2:



$$f_n(x) = \begin{cases} 2n^2 x, & 0 \leq x \leq \frac{1}{2n} \\ -2n^2\left(x - \frac{1}{n}\right), & \frac{1}{2n} < x \leq \frac{1}{n} \\ 0, & \text{otherwise} \end{cases}$$

One has $f_n(x) \xrightarrow{n \to \infty} 0 \ \forall x$ fixed,

and $0 \in L^1([0,1]) \ni f_n$, but $\|f_n - 0\|_1 = \|f_n\|_1 = \frac{1}{2}$.

(convergent pointwise but not in L^1-norm)

Def. For any $p \geq 1$ we set $\quad p \in \{p \in \mathbb{R} \mid p \geq 1\}$

$$L^p([a,b]) := \{f \text{ measurable on } [a,b] \mid |f|^p \text{ is L. integrable}\} / \sim$$

We define for $f \in L^p([a,b])$

$$\|f\|_p := \left[\int_a^b |f(x)|^p \, dx\right]^{1/p}$$

Prop. $L^p([a,b])$ with $\|\bullet\|_p$ is a Banach space.

$$\Rightarrow \|f+g\|_p \leq \|f\|_p + \|g\|_p$$

Thm. (Hölder inequality)

If $f \in L^p([a,b])$ and $g \in L^q([a,b])$ with $p \geq 1$, $\frac{1}{p} + \frac{1}{q} = 1$

Then $fg \in L^1([a,b])$ and

$$\|fg\|_1 \leq \|f\|_p \|g\|_q$$

Trick for proof:

$\forall a, b > 0$ and $\alpha, \beta \in (0,1)$ with $\alpha + \beta = 1$:

$$ab \leq \alpha a^{1/\alpha} + \beta b^{1/\beta}$$

Remark: $L^\infty([a,b])$ with $\|f\|_\infty := \operatorname*{ess\,sup}_{x \in [a,b]} |f(x)|$ is again a Banach space.

Remark: For $p = 2$, $L^2([a,b])$ has also an inner product

$$\langle f, g \rangle := \int_a^b \overline{f}(x) \, g(x) \, dx$$

By Hölder inequality,

$$|\langle f, g \rangle| = \|fg\|_1 \leq \|f\|_2 \|g\|_2$$

Remark: extension naturally to $L^p([a, \infty))$ and $L^p(\mathbb{R})$.