3) Lebesgue integration (on $\mathbb{R}$)

Aim: to define an integration more general than the Riemannian integration, which gives the same value for Riemann integrable functions.

Def. Let $f:[a,b] \longmapsto \mathbb{R}$. f is Lebesgue measurable on $[a,b]$ if $\forall s \in \mathbb{R}: \{x \in [a,b] \mid f(x) > s\}$ is Lebesgue measurable.

$\hookrightarrow = f^{-1}((s,\infty))$

Lemma: The set of Lebesgue measurable functions on $[a,b]$ is a vector space (they can be added) and is an algebra (they can be multiplied) and $\frac{f}{g}$ is also L. measurable if f, g are L. measurable and $g(x) \neq 0$.

Proof as a (not so easy) exercise.

Def. Let $f, g: [a,b] \longmapsto \mathbb{R}$.

1) $f = g$ a.e. (= almost everywhere) if $\{x \in [a,b] \mid f(x) \neq g(x)\}$ is Lebesgue measurable and has measure 0.

2) $f \leq g$ a.e. if $\{x \in [a,b] \mid f(x) > g(x)\}$ is Lebesgue measurable and has measure 0.

Proposition

Let $f, g: [a,b] \longmapsto \mathbb{R}$ with $f = g$ a.e.

Then if f is L. measurable then g is also L. measurable.

Additional important notions

Let $\{f_n\}_{n \in \mathbb{N}}$ be a pointwise bounded sequence of functions defined on $[a,b]$, it means $\forall x \in [a,b]: \{f_n(x)\}_{n \in \mathbb{N}} \subset \mathbb{R}$ is bounded.

Then for fixed $x \in [a,b]$, we set

$$f^*(x) = \limsup_{n\to\infty} f_n(x) := \lim_{n\to\infty} \underbrace{\sup_{m \geq n} \{f_m(x)\}}_{\text{decreasing with } n \text{ and bounded below}}$$ always well-defined

and

$$f_*(x) = \liminf_{n\to\infty} f_n(x) := \lim_{n\to\infty} \inf_{m \geq n} \{f_m(x)\}$$

⚠ These 2 notations are always well-defined! ⤳ useful

Observe that $f_*(x) \leq f^*(x)$ and that $f_*(x) = f^*(x)$ iff $\lim_{n\to\infty} f_n(x)$ exists.

Thm. If each f_n is Lebesgue measurable on $[a,b]$ and $\{f_n\}_n$ is pointwise bounded then $f^*: [a,b] \longmapsto \mathbb{R}$ and $f_*: [a,b] \longmapsto \mathbb{R}$ are Lebesgue measurable.

Corollary: If $\{f_n\}$ a sequence of L.m. functions on $[a,b]$, and if $\exists f: [a,b] \longmapsto \mathbb{R}: \lim_{n\to\infty} f_n(x) = f(x)$ a.e.

Then f is L. measurable.

useful for the Lebesgue dominated convergence thm

This notion is useful for defining the Lebesgue integral, of f, by a family of approximation (f_n). In particular, we can consider

$$f_n = \sum_{k=1}^{n} c_k \chi_{E_k}$$ with $c_k \in \mathbb{R}$ and E_k a measurable subset of $[a,b]$.

check that f_n is L. measurable.

$$\chi_{E_k}(x) := \begin{cases} 1 & \text{if } x \in E_k \\ 0 & \text{if } x \notin E_k \end{cases}$$

Def. A measurable partition ~~of~~ $\mathcal{P}$ of $[a,b]$ consists in a finite collection $\{E_j\}_{j=1}^{N} \subseteq [a,b]$ s.t.

- E_j is L. m.
- $\bigcup_j E_j = [a,b]$
- $m(E_j \cap E_k) = 0 \quad \forall j \neq k$

⚠ These partitions are much more general than the ones seen in Calculus I.

For $f \in L^\infty([a,b])$ and for $\mathcal{P}$ a measurable partition of $[a,b]$ we set

$$U(f,\mathcal{P}) := \sum_{j=1}^{N} \left(\sup_{x \in E_j} f(x) \right) m(E_j)$$

$$L(f,\mathcal{P}) := \sum_{j=1}^{N} \left(\inf_{x \in E_j} f(x) \right) m(E_j)$$

Def. f is Lebesgue integrable if

$$\sup_{\mathcal{P}} L(f,\mathcal{P}) = \inf_{\mathcal{P}} U(f,\mathcal{P})$$

and if so, we write

$$\int_a^b f(x)\,dx := \sup_{\mathcal{P}} L(f,\mathcal{P}) = \inf_{\mathcal{P}} U(f,\mathcal{P})$$

Remark:

If $f(x) = \begin{cases} 1 & \text{if } x \in [0,1] \cap \mathbb{Q} \\ 0 & \text{if } x \in [0,1] \setminus \mathbb{Q} \end{cases}$

then f is L. integrable and $\int_0^1 f(x)\,dx = 0$. (Exercise)

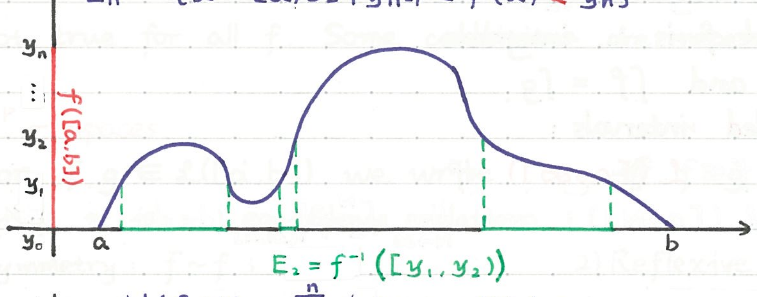
⚠ This function was not Riemannian integrable.

Thm. If $f \in L^\infty([a,b])$ is Riemannian integrable then f is Lebesgue integrable with the same integral.

Thm. Let $f \in L^\infty([a,b])$. Then f is Lebesgue integrable iff f is Lebesgue measurable. Not trivial proof

Remark 1: If f is Lebesgue measurable on $[a,b]$, consider a partition of $f([a,b])$.
namely $y_0 = \inf f([a,b]) < y_1 < y_2 < \cdots < y_n = \sup f([a,b])$
and set $E_j := \{x \in [a,b] \mid y_{j-1} \leq f(x) < y_j\}$ L. measurable since f is L.m. for $j = 0, 1, \cdots, n-1$

⚠ $E_n = \{x \in [a,b] \mid y_{n-1} \leq f(x) \leq y_n\}$



Consider $U(f,Y) = \sum_{j=1}^{n} y_j \, m(E_j)$;

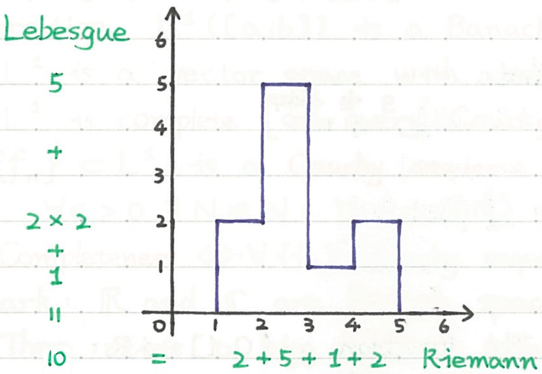
$\{E_j\}_{j=1}^{n} =:$

$L(f,Y) = \sum_{j=1}^{n} y_{j-1} \, m(E_j)$

By taking finer partitions of $f([a,b])$ we converge to $\int_a^b f(x)\,dx$!

Remark 2: We can define $\sum_{j=1}^{n} y_j \chi_{E_j}$ and a family of such approximations converging to f.

Picture:



Remark:

1) Suppose $f \geq 0$ but not bounded from above. Then we can consider $N \in \mathbb{N}$,

$$f_N(x) = \begin{cases} f(x), & \text{if } f(x) < N \\ N, & \text{if } f(x) \geq N \end{cases} \qquad (*)$$

Then if f is L.m. on $[a,b]$, f_N is also L.m. on $[a,b]$ and we consider
$$\int_a^b f_N(x)\,dx \xrightarrow{N\to\infty} \int_a^b f(x)\,dx$$
if this limit exists.

2) If $f \leq 0$ but not bounded from below, Replace N by $(-N)$ in $(*)$

3) If f is not of a definite sign and unbounded from below and above, we need to consider $f = f_+ - f_-$ with $f_+ \geq 0$ and $f_- \geq 0$ and we do separately f_+ and f_-.