Lebesgue Theory

1) Reminder on Riemann integration

Let $\mathcal{P}$ be a partition of $[a,b]$. It means

$$\mathcal{P} = \{x_0, x_1, \cdots, x_n\} \text{ with } x_0 = 0,\ x_n = b,\ x_j < x_{j+1}.$$

Let $f \in \mathcal{L}^\infty([a,b])$, it means

$$f : [a,b] \longmapsto \mathbb{R} \qquad \text{with } \sup_{x\in[a,b]} |f(x)| \equiv \|f\|_\infty < \infty$$

One sets

$$\underset{\substack{\uparrow \\ \text{lower}}}{L}(f,\mathcal{P}) = \sum_j \inf_{x\in[x_i,x_j]} f(x)\,(x_j - x_{j-1}) \geq (b-a) \underbrace{\inf_{x\in[a,b]} f(x)}_{:= m}$$

$$U(f,\mathcal{P}) = \sum_j \sup_{x\in[x_i,x_j]} f(x)\,(x_j - x_{j-1}) \leq (b-a) \underbrace{\sup_{x\in[a,b]} f(x)}_{:= M}$$

We define the lower integral by $\sup\limits_{\mathcal{P}} L(f,\mathcal{P})$, $\leq$ Exercise: check
and the upper integral by $\inf\limits_{\mathcal{P}} U(f,\mathcal{P})$.

Def. f is Riemann integrable if $\sup\limits_{\mathcal{P}} L(f,\mathcal{P}) = \inf\limits_{\mathcal{P}} U(f,\mathcal{P})$,
and we write $\int_a^b f(x)\,dx :=$

Remark: Consider the function $f : [0,1] \longmapsto \mathbb{R}$:

$$f(x) = \begin{cases} 1 & \text{if } x \in \mathbb{Q} \\ 0 & \text{if } x \in \mathbb{R}\setminus\mathbb{Q} \end{cases}$$

Observe that $L(f,\mathcal{P}) = 0$ and $U(f,\mathcal{P}) = 1 \ \forall \mathcal{P}$.

$\Rightarrow f$ is not Riemann integrable.

Theorem: If $f \in C([a,b],\mathbb{R})$ then f is Riemann integrable.

Main ingrediant: use that f is uniformly continuous on $[a,b]$.
$\uparrow \forall \varepsilon > 0\ \exists \delta$ (indep of x) ...

We would like to have a more general theory of integration.

2) Lebesgue measure

Reminder: A subset $V \subset \mathbb{R}^n$ is open if

$$\forall x \in V\ \exists \varepsilon > 0 : B(x,\varepsilon) \subset V$$

$\hookrightarrow$ open ball centred at x with radius ε

V is closed in $\mathbb{R}^n$ if $\mathbb{R}^n \setminus V$ is open.

V is compact in $\mathbb{R}^n$ if V is closed and bounded

$\hookrightarrow \exists R > 0 : V \subset B(0,R)$

Def. A closed box in $\mathbb{R}^n$ is a set I of the form

$$I := \{(x_1, \cdots, x_n) \in \mathbb{R}^n \mid a_i \leq x_i \leq b_i \ \forall i \in \{1, \cdots, n\}\ \text{for some fixed } a_i, b_i \in \mathbb{R}\}$$

For $n = 1$  ; For $n = 2$ 

The volume of I is given by $\mathrm{vol}(I) := \prod_{j=1}^{n} (b_j - a_j)$.

For more general sets in $\mathbb{R}^n$, we use approximation procedure.

Def. For any set $V \subset \mathbb{R}^n$, the set

$S := \{I_j\}_j$ closed box for any j is a covering of V if $V \subset \bigcup_j I_j$.

S can be finite or countably infinite. We set

$\sigma(S) := \sum_j \mathrm{vol}(I_j) \in [0, \infty]$

Def. For any $V \subset \mathbb{R}^n$ the Lebesgue outer measure of V is

$m^*(V) := \inf \{\sigma(S) \mid S \text{ a covering of } V\}$

Exercise: $m^*(\underset{\text{closed box}}{I}) = \mathrm{vol}(I)$

Prop.

1) If $V \subset W \subset \mathbb{R}^n$, then $m^*(V) \leq m^*(W)$
2) $m^*(V \cup W) \leq m^*(V) + m^*(W)$
3) $m^*(\bigcup_j V_j) \leq \sum_j m^*(V_j)$

⚠ Even if $V_1 \cap V_2 = \phi$ one can have $m^*(V_1 \cup V_2) < m^*(V_1) + m^*(V_2)$ not desirable

Exercise: find examples.

Thm. Let $V \subset \mathbb{R}^n$ with $m^*(V) < \infty$.

Then for any $\varepsilon > 0$, $\exists W$ open with $V \subset W$ and $m^*(W) \leq m^*(V) + \varepsilon$. W is only slightly bigger

If $m^*(V) = \infty$, the statement is true for $W = \mathbb{R}^n$.

Note that from the Thm and the Property 2,

$m^*(W) \leq m^*(V) + m^*(W \setminus V)$ for $W = V \sqcup (W \setminus V)$

$\Rightarrow m^*(W) - m^*(V) \leq m^*(W \setminus V)$ lower bound

But can we get an upper estimate for $m^*(W \setminus V)$?

Def. A set $V \subset \mathbb{R}^n$ is Lebesgue measurable if

if for any $\varepsilon > 0$, $\exists W$ open with $V \subset W \subset \mathbb{R}^n$ such that $m^*(W \setminus V) \leq \varepsilon$.

For Lebesgue measurable sets, we set the Lebesgue measure as

$m(V) := m^*(V)$

⚠ It means not all sets are Lebesgue measurable.

Prop. (proof as exercise)

- If V is open, then V is Lebesgue measurable.
- If $m^*(V) = 0$, then V is Lebesgue measurable.
- If $V = \bigcup_j V_j$ with V_j L.m., then V is L.m. with $m(V) \leq \sum_j m(V_j)$
- If $V = \bigcap_j V_j$ with V_j L.m., then V is L.m.

- If V is closed, then V is L.m.

  In particular, any closed box is L.m., and $m(I) = vol(I)$
  ↖ closed box

Thm.

If $\{V_j\}$ is a countable family of L.m. sets with pairwise empty intersecti

then $m(\bigcup_j V_j) = \sum_j m(V_j)$.

This was not true for m^*.

Remark: There exist sets which are not L.m.

Example: On $\mathbb{R}$ we consider the equivalence relation

$x \sim y$ if $x - y \in \mathbb{Q}$

We denote by $[x]$ the equivalence class containing x. E.g.

$[\sqrt{2}] \neq [\sqrt{3}] \neq [0] = \mathbb{Q}$

Consider $V \subset \mathbb{R}$ made of 1 representative of each equivalence class. Axiom of choice

Then V is NOT L. measurable.

Report about this, or about G_δ sets. (see the book) §1.3