Prop. $\mathcal{F}$ extends to a bijective map from $L^2(\mathbb{R}^n)$ to $L^2(\mathbb{R}^n)$
$(f \in L^2(\mathbb{R}^n)$ iff $\int |f(x)|^2 dx < \infty)$ and

$$[\mathcal{F}^{-1} f](x) = \frac{1}{(2\pi)^{n/2}} \int_{\mathbb{R}^n} e^{+ix\cdot y} f(y)\, dy$$

and

$$\|f\|_{L^2(\mathbb{R}^n)} = \|\mathcal{F} f\|_{L^2(\mathbb{R}^n)} \quad \text{(Plancherel equality)}$$

$$\|f\|_{L^2(\mathbb{R}^n)} = \left(\int_{\mathbb{R}^n} |f(x)|^2 dx\right)^{\frac{1}{2}}$$

⚠ $\int e^{-ix\cdot y} f(y)\, dy$ for $f \in L^2(\mathbb{R}^n)$ is not well-defined. We have to consider
$\lim_{R\to\infty} \int_{[-R,R]^n} e^{-ix\cdot y} f(y)\, dy$

We would like to define $\mathcal{F}T$ (Fourier transform for a distribution) for $T \in \mathcal{D}'(\mathbb{R}^n)$, and the normal expression should be $[\mathcal{F}T](\varphi) = T(\mathcal{F}\varphi)$.

Problem: For $\varphi \in \mathcal{D}(\mathbb{R}^n)$, $\mathcal{F}\varphi \in \underbrace{C_0(\mathbb{R}^n)}_{:= \{f \in C(\mathbb{R}^n) \mid \lim_{|x|\to\infty} f(x) = 0\}}$ but $\mathcal{F}\varphi \notin \mathcal{D}(\mathbb{R}^n)$ in general.

We need to replace $\mathcal{D}(\mathbb{R}^n)$ by a set invariant under Fourier transform.

Def. Let $\mathcal{S}(\mathbb{R}^n)$ be the subset of $C^\infty(\mathbb{R}^n)$ such that

$$\sup_{x\in\mathbb{R}^n} |x^\beta \partial^\alpha f(x)| < \infty \quad \forall f \in \mathcal{S}(\mathbb{R}^n),\ \alpha, \beta \in \mathbb{N}^n$$

$\forall x \in \mathbb{R}^n, \beta \in \mathbb{N}^n$: $x^\beta := x_1^{\beta_1} x_2^{\beta_2} \cdots x_n^{\beta_n}$

$\mathcal{S}(\mathbb{R}^n)$ is called the Schwartz space.

Clearly

$$\mathcal{D}(\mathbb{R}^n) \subset \mathcal{S}(\mathbb{R}^n) \subset C^\infty(\mathbb{R}^n) \subset \underset{\substack{\rotatebox{90}{$\in$}\\ h}}{L^1_{loc}}(\mathbb{R}^n) \subset \underset{\substack{\rotatebox{90}{$\in$}\\ T_h}}{\mathcal{D}'}(\mathbb{R}^n)$$

$h \longleftrightarrow T_h$

Properties:

- $\mathcal{S}(\mathbb{R}^n)$ is a vector space.
- If $f \in \mathcal{S}(\mathbb{R}^n)$, then $x^\beta \partial^\alpha f \in \mathcal{S}(\mathbb{R}^n)$.
- $\mathcal{F}\mathcal{S}(\mathbb{R}^n) = \mathcal{S}(\mathbb{R}^n)$ easy

Def. (Convergence in $\mathcal{S}(\mathbb{R}^n)$)
A sequence $(f_j)_j \subset \mathcal{S}(\mathbb{R}^n)$ converges to $f_\infty \in \mathcal{S}(\mathbb{R}^n)$ if

$$\sup_{x\in\mathbb{R}^n} |x^\beta \partial^\alpha (f_j - f_\infty)(x)| \xrightarrow{j\to\infty} 0 \quad \forall \alpha, \beta \in \mathbb{N}^n$$

⚠ Not the same notation as in $\mathcal{D}(\mathbb{R}^n)$!

Def. A tempered distribution is a distribution which is linear and continuous on $\mathcal{S}(\mathbb{R}^n)$.
It means if $f_j \xrightarrow{j\to\infty} f_\infty$ in $\mathcal{S}(\mathbb{R}^n)$ then $T(f_j) \xrightarrow{j\to\infty} T(f_\infty)$.
The set of tempered distributions is denoted by $\mathcal{S}'(\mathbb{R}^n)$.
Clearly $\mathcal{S}'(\mathbb{R}^n) \subset \mathcal{D}'(\mathbb{R}^n)$ and observe that $\mathcal{D}'(\mathbb{R}^n) \ni T_{x\mapsto e^{x^2}} \notin \mathcal{S}'(\mathbb{R}^n)$.
Indeed $T_{x\mapsto e^{x^2}}(\varphi) = \int e^{x^2} \varphi(x)\, dx$ not well defined for $\varphi \in \mathcal{S}(\mathbb{R}^n)$

$L^1_{loc}(\mathbb{R}^n) \not\subset S'(\mathbb{R}^n)$ since $[x \mapsto e^{x^2}] \in L^1_{loc}(\mathbb{R}^n)$ but $T_{x\mapsto e^{x^2}} \notin S'(\mathbb{R}^n)$.

Observe that if $T \in S'(\mathbb{R}^n)$ then $\partial^\alpha T \in S'(\mathbb{R}^n)$

since $\partial^\alpha \varphi \in S(\mathbb{R}^n)$ if $\varphi \in S(\mathbb{R}^n)$; $(\partial^\alpha T)(\varphi) = (-1)^\alpha T(\partial^\alpha \varphi)$ and one can take care of the continuity.

Prop. The distribution T is tempered iff :

$\exists c > 0, m \in \mathbb{N}$:

$$|\langle T, \varphi \rangle| \equiv |T(\varphi)| \leq c \sum_{|\alpha|, |\beta| \leq m} \sup_{x \in \mathbb{R}^n} |x^\beta \partial^\alpha \varphi(x)| \quad \forall \varphi \in S(\mathbb{R}^n).$$

No localization compared to the creterion for $T \in \mathcal{D}'(\mathbb{R}^n)$ (p. 3).

Def. For any $T \in S'(\mathbb{R}^n)$ we set

$$[\mathcal{F}T](\varphi) := T(\underbrace{\mathcal{F}\varphi}_{\in S(\mathbb{R}^n)}) \quad \forall \varphi \in S(\mathbb{R}^n)$$

⚠ The Fourier transform is not defined for all distributions but on the tempered distributions.

Observation: $\mathcal{F}S'(\mathbb{R}^n) = S'(\mathbb{R}^n)$

(exercise) ↑ follows from $\mathcal{F}S(\mathbb{R}^n) = S(\mathbb{R}^n)$

Examples:

1) $\mathcal{F}\delta_0 = T_{(\frac{1}{2\pi})^{n/2}\cdot 1}$ } constant function

Proof:

$$[\mathcal{F}\delta_0](\varphi) = \delta_0(\mathcal{F}\varphi) = \delta_0\left(x \mapsto \frac{1}{(2\pi)^{n/2}} \int e^{-ix\cdot y} \varphi(y)\, dy\right)$$
$$= \frac{1}{(2\pi)^{n/2}} \int_{\mathbb{R}^n} \varphi(y)\, dy$$
$$= \int_{\mathbb{R}^n} \frac{1}{(2\pi)^{n/2}} \cdot 1\, \varphi(y)\, dy$$
$$= T_{(\frac{1}{2\pi})^{n/2}\cdot 1}(\varphi) \qquad \forall \varphi \in S(\mathbb{R}^n)$$

(Often written as $\mathcal{F}\delta_0 = \frac{1}{(2\pi)^{n/2}}$ but a little misleading)

2) $\mathcal{F}T_1 = (2\pi)^{n/2}\delta_0$

3) $\mathcal{F}T_h = T_{\hat{h}}$ for $h \in L^2(\mathbb{R}^n)$

↑ $\hat{h} := \mathcal{F}(h)$

5) Continuous extensions

Def. A distribution $T \in \mathcal{D}'(\mathbb{R}^n)$ is summable if

$\exists\, m \geq 0$ and $c > 0$:

$$|T(\varphi)| \leq c \sum_{|\alpha| \leq m} \|\partial^\alpha \varphi\|_\infty \quad \forall \varphi \in \mathcal{D}(\mathbb{R}^n)$$

No localization (m, c are global)

The smallest m satisfying this is called the summability order of T

(denoted sum-order $(T) = m$)

Remark: The space of summable distributions with sum-order 0 coincides with the set of bounded measures on $\mathbb{R}^n$

(denoted by $M_b(\mathbb{R}^n)$)

Remarks:

sum-order $(\delta_0) = 0$;

sum-order $(\delta_0^\alpha) = |\alpha|$

For any $m \in \mathbb{N}$, we define

$$C_b^m(\mathbb{R}^n) := \{ f \in C^m(\mathbb{R}^n) \mid \partial^\alpha f \text{ is bounded for } \|\alpha\| \leq m \}$$

$C_b^m(\mathbb{R}^n)$ is endowed with a norm

$$\|f\|_{C_b^m(\mathbb{R}^n)} = \sum_{\|\alpha\| \leq m} \|\partial^\alpha f\|_\infty$$

and $C_b^m(\mathbb{R}^n)$ is complete with this norm.

Remark: $\mathcal{D}(\mathbb{R}^n) \subset \mathcal{S}(\mathbb{R}^n) \subset C_b^m(\mathbb{R}^n)$

Lemma: If T is summable with sum-order m

then T can be (continuously) extended

to a continuous linear functional on $C_b^m(\mathbb{R}^n)$

$$T : C_b^m(\mathbb{R}^n) \longrightarrow \mathbb{C}$$

In summary: even if, for the general theory, one starts with $\mathcal{D}(\mathbb{R}^n)$ a very small set of function
we can always extend the validity of a given distribution
to a much larger set.