2) Derivatives of distributions

Extension: support of a distribution

Recall that for $f, g : \mathbb{R} \mapsto \mathbb{R}$,

$$\int_a^b f'(x) g(x)\, dx = f(x) g(x) \Big|_a^b - \int_a^b f(x) g'(x)\, dx$$

If $a = -\infty$, $b = +\infty$, and $f, g \in \mathcal{D}(\mathbb{R})$, then

$$\int_a^b f'(x) g(x)\, dx = - \int_a^b f(x) g'(x)\, dx$$

Similarly, if $f, g \in \mathcal{D}(\mathbb{R}^n)$,

$$\int_{\mathbb{R}^n} (\partial_j f)(x)\, g(x)\, dx = - \int_{\mathbb{R}^n} f(x)\, (\partial_j g)(x)\, dx$$

By induction, one gets

$$\int_{\mathbb{R}^n} (\partial^\alpha f)(x)\, g(x)\, dx = (-1)^{|\alpha|} \int_{\mathbb{R}^n} f(x)\, (\partial^\alpha g)(x)\, dx$$

Def. For $T \in \mathcal{D}'(\mathbb{R}^n)$ and for $f \in \mathcal{D}(\mathbb{R}^n)$ one sets

$$(\partial^\alpha T)(f) := (-1)^{|\alpha|}\, T(\partial^\alpha f)$$

Remark: We want to have

$$(\partial^\alpha T_h)(f) = (-1)^{|\alpha|}\, T_h(\partial^\alpha f) = (-1)^{|\alpha|} \int_{\mathbb{R}^n} h(x) (\partial^\alpha f)(x)\, dx$$
$$= \int_{\mathbb{R}^n} (\partial^\alpha h)(x) f(x)\, dx = T_{\partial^\alpha h}(f)$$

if h is regular enough.

Lemma: $\partial^\alpha T \in \mathcal{D}'(\mathbb{R}^n)$

Exercise: proof ⤴ (use the prop of last week)

Remark: In particular, T_h can be diffrenciated any times for any $h \in L^1_{loc}(\mathbb{R}^n)$

It means, any function in $L^1_{loc}(\mathbb{R}^n)$ is differenciable in the sense of distribution.

Examples:

1) Let $H : \mathbb{R} \mapsto \mathbb{R}$ be the Heaviside function:

$$H(x) := \begin{cases} 0 & \text{if } x < 0 \\ 1 & \text{if } x \geq 0 \end{cases}$$

Clearly $H \in L^1_{loc}(\mathbb{R}) \Rightarrow T_H \in \mathcal{D}'(\mathbb{R})$. What is ∂T_H?

$$(\partial T_H)(f) = - T_H(\partial f) = - T_H(f') = - \int_{\mathbb{R}} H(x) f'(x)\, dx = - \int_0^\infty f'(x)\, dx$$
$$= \underbrace{-f(\infty)}_{=0} + f(0) = f(0) = \delta_0(f)$$

$\Rightarrow \partial T_H = \delta_0$ (Horribly written as $\partial H = \delta_0$)

2) Recall that $P_v \frac{1}{x} \in \mathcal{D}'(\mathbb{R})$ (principal value) defined by

$$(P_v \tfrac{1}{x})(f) := \lim_{\varepsilon \searrow 0} \Big[\int_{-\infty}^{-\varepsilon} \tfrac{1}{x} f(x)\, dx + \int_{\varepsilon}^{\infty} \tfrac{1}{x} f(x)\, dx \Big]$$

$\tfrac{1}{x} = \ln(|x|)'$

Exercise: ↳ $\in \mathcal{D}'(\mathbb{R})$

1)

2)

3) $\ln(x) \in L^1_{loc}(\mathbb{R})$

$$= \lim_{\varepsilon \searrow 0} \Big[\int_{-\infty}^{-\varepsilon} \ln(|x|)' f(x)\, dx + \int_{\varepsilon}^{\infty} \ln(|x|)' f(x)\, dx \Big]$$

$$= - \int_{\mathbb{R}} \ln(|x|) f'(x)\, dx = \partial T_{\ln(|x|)}(f)$$

$\Rightarrow \partial T_{\ln(|x|)} = P_v \frac{1}{x}$ (Also $\partial \ln(|\cdot|) = P_v \frac{1}{x}$)

Extension: For $n=3$ consider the function

$$h : \mathbb{R}^3 \setminus \{0\} \ni x \longmapsto \frac{1}{\|x\|} \in \mathbb{R}$$

and observe that $h \in L^1_{loc}(\mathbb{R}^3) \Rightarrow T_h \in \mathcal{D}'(\mathbb{R}^3)$

Recall that

$$\Delta = \partial_1^2 + \partial_2^2 + \partial_3^2$$

Question: what is $\Delta T_h = -4\pi\delta_0$?

(in Physics $\Delta \frac{1}{\|x\|} = -4\pi \delta_0^3$) ← for dim

Ref : Section 3.5 of the reference book

3) Other operations with distribution

Def.

- **Multiplication by smooth function :**

  Let $T \in \mathcal{D}'(\mathbb{R}^n)$ and $g \in C^\infty(\mathbb{R}^n)$. We set

  $$(gT)(f) := T(\underbrace{gf}_{\in \mathcal{D}(\mathbb{R}^n)}) \quad \text{for } f \in \mathcal{D}(\mathbb{R}^n)$$

Exercise: check that $gT \in \mathcal{D}'(\mathbb{R}^n)$.

Remark: If $T = T_h$ then $gT_h = T_{\underbrace{gh}_{\in L^1_{loc}(\mathbb{R}^n)}}$ if $h \in L^1_{loc}(\mathbb{R}^n)$

Examples:
- $g\delta_0 = g(0)\delta_0$
- $g\delta_0' = g(0)\delta_0' - g'(0)\delta_0$ (exercise)

- **Convergence in $\mathcal{D}'(\mathbb{R}^n)$**

  A sequence $\{T_i\}_{i\in\mathbb{N}}$ converges to $T_\infty \in \mathcal{D}'(\mathbb{R}^n)$ if

  $$T_i(f) \longrightarrow T_\infty(f) \quad \forall f \in \mathcal{D}(\mathbb{R}^n)$$

Remark: If $\{T_i(f)\}$ converges in $\mathbb{C}$ or in $\mathbb{R}$, for any $f \in \mathcal{D}(\mathbb{R}^n)$ then $\exists T_\infty \in \mathcal{D}(\mathbb{R}^n) : T_i \xrightarrow{i\to\infty} T_\infty$ } completeness

Remark: If $T_i \xrightarrow{i\to\infty} T_\infty$ in distribution, then $\partial^\alpha T_i \xrightarrow{i\to\infty} \partial^\alpha T_\infty$ in distribution.

Example:

Set $h_i : \mathbb{R} \longrightarrow \mathbb{R}$, $h_i(x) = \dfrac{\sin(ix)}{x}$, $i \in \mathbb{N}$

Then $T_{h_i} \xrightarrow{i\to\infty} \pi\delta_0$ in distribution.

- Convolution of distributions

If $f, g : \mathbb{R} \mapsto \mathbb{R}$ we set

$$(f*g)(x) := \text{const} \int f(x-y)\, g(y)\, dy$$

for the convolution of f and g.

Extension: sometimes we can also define the convolution of 2 distributions.

Remark: Every distribution can be written locally as $\partial^\alpha T_h$ for some $h \in C(\mathbb{R}^n)$ and some $\alpha \in \mathbb{N}^n$.

(Take $R>0$ and $f \in \mathcal{D}(\mathbb{R}^n)$ with $\operatorname{supp}(f) \in B(0,R)$
Then $T(f) \equiv \partial^\alpha T_h(f)$ $\exists\, \alpha \in \mathbb{N}^n$ and $h \in C(\mathbb{R}^n)$)

4) Fourier transform, Schwartz functions and tempered distributions.

Def. For any $f \in L^1(\mathbb{R}^n) := \{ f : \mathbb{R}^n \mapsto \mathbb{R} \mid \int_{\mathbb{R}^n} |f(x)|\, dx < \infty \}$ we set

scalar product in $\mathbb{R}^n$

$$[\mathcal{F}f](x) \equiv \hat{f}(x) := \frac{1}{(2\pi)^{n/2}} \int_{\mathbb{R}^n} e^{-ix\cdot y} f(y)\, dy$$

We call it the Fourier transform of f.

Properties:

- $\mathcal{F}$ is linear
- $|\hat{f}(x)| \leq \int |f(y)|\, dy < \infty$
- $\hat{f} \in C_0(\mathbb{R}^n) := \{ g \in C(\mathbb{R}^n) \mid \lim_{\|x\| \to \infty} g(x) = 0 \}$
- $\mathcal{F}(f*g) = \hat{f} \cdot \hat{g}$ $\left(\text{const} = \frac{1}{(2\pi)^{n/2}}\right)$
- If $f, \partial_j f \in L^1(\mathbb{R}^n)$ then $\widehat{-i\partial_j f}(x) = x_j \hat{f}(x)$

with $i^2 = -1$