III.2 Spectral parts of a self-adjoint operators

Reminders about measures on $\mathbb{R}$: 3 types of measures

1) m is a pure point measure if
$$\forall v \in \mathcal{A}_B \ \exists \{x_j\}_j \subset v : \ m(v) = \sum_j m(\{x_j\})$$

2) m is an absolutely continuous measure with respect to the Lebesgue measure if
$$\forall v \in \mathcal{A}_B : m(v) = \int_v f(x)\,dx \quad \text{for some } f \in L^1(v).$$
dx always means Lebesgue integral.
f = Radon Nykodym density ; f does not depend on v.

3) m is a singular continuous measure with respect to the Lebesgue measure if
$$\forall x \in \mathbb{R} : m(\{x\}) = 0 \quad \text{and}$$
$$\exists v \in \mathcal{A}_B : \int_v 1\,dx = 0 \quad \text{and} \quad m(\mathbb{R}\setminus v) = 0$$
It means the support of m is concentrated on a set of Lebesgue measure 0, but which is not countable.

Thm. (Lebesgue decomposition thm)
Any Stieltjes (= regular Borel) measure m admits a unique decomposition
$$m = m_{pp} + m_{ac} + m_{sc}$$
with m_{pp} pure point
$$\left.\begin{array}{ll} m_{ac} & \text{absolutely continuous} \\ m_{sc} & \text{singular continuous} \end{array}\right\} \text{with respect to the Lebesgue measure.}$$

Remark: The measures $m_f = \langle f, E(\cdot) f \rangle$ are Stieltjes measures!
$f \in \mathcal{H}$; spectral measure

Let A be a self-adjoint op. and let $\{E^A_\lambda\}_{\lambda\in\mathbb{R}}$ be the associated spectral family, and E^A the associated spectral measure.
Recall that
$$\mathrm{Ran}\,(E^A(\{\mu\})) = \{f \in \mathcal{H} \mid E^A(\{\mu\}) f = f\}$$
Observe that
$$\mathrm{Ran}\,(E^A(\{\mu\})) = \{f \in D(A) \mid Af = \mu f\}$$
Indeed this follows from the equality
$$\|(A-\mu)f\|^2 = \int |\lambda-\mu|^2\, m_f(d\lambda)$$
If f satisfies $Af = \mu f$, then l.h.s $= 0 \iff m_f$ is only supported at μ.

Def. $\mathcal{H}_p(A) := \bigoplus_{\mu} \mathrm{Ran}\,(E^A(\{\mu\})) \subset \mathcal{H}$
μ: eigenvalues of A — this set is countable (because $\mathcal{H}$ is seperable)

$\mathcal{H}_p(A)$ is the subspace generated by all eigenfunctions for all eigenvalues of A.

Remark: This subset can be small, but for matrices it is all the finite dimensional space.

Def. We set

$\mathcal{H}_{ac}(A) := \{f \in \mathcal{H} \mid m^A_f \text{ is a. c.}\}$

$\mathcal{H}_{sc}(A) := \{f \in \mathcal{H} \mid m^A_f \text{ is s. c.}\}$

Thm. (Decomposition thm)

$$A = \begin{pmatrix} A_p & 0 & 0 \\ 0 & A_{ac} & 0 \\ 0 & 0 & A_{sc} \end{pmatrix}$$

These blocks are either 0 or infinite dimensional.

1) $\mathcal{H}_p(A) = \{f \in \mathcal{H} \mid m^A_f \text{ is pure point}\}$

2) $\mathcal{H} = \mathcal{H}_p(A) \oplus \mathcal{H}_{ac}(A) \oplus \mathcal{H}_{sc}(A)$ and
$A = A_p \oplus A_{ac} \oplus A_{sc}$ which are self-adjoint restrictions.

3) If $\varphi : \mathbb{R} \to \mathbb{C}$ is bounded and continuous,
$\varphi(A) = \varphi(A_p) \oplus \varphi(A_{ac}) \oplus \varphi(A_{sc})$

4) $\sigma(A) = \underbrace{\sigma(A_p)}_{= \overline{\sigma_p(A)}} \cup \sigma(A_{ac}) \cup \sigma(A_{sc})$ closure

These thms are not so difficult; they take some time.

Def. (2 more sets)

$\sigma_d(A) = \{\lambda \in \sigma_p(A) \mid \lambda \text{ is isolated in } \sigma(A) \text{ and of finite multiplicity}\} \subset \mathbb{R}$

discrete

$\hookrightarrow \exists \varepsilon > 0 : (\lambda - \varepsilon, \lambda + \varepsilon) \cap \sigma(A) = \{\lambda\}$

$\sigma_{ess}(A) = \sigma(A) \setminus \sigma_d(A) \subset \mathbb{R}$

essential

Stability of the essential spectrum important for perturbation theory

Prop. Let A be self-adjoint, B be symmetric and such that

$B(A-z)^{-1} \in \mathcal{K}(\mathcal{H})$ B is A-compact for $z \in \mathbb{C} \setminus \mathbb{R}$

Then

$$\sigma_{ess}(A+B) = \sigma_{ess}(A)$$

The nature of the spectrum can change a lot !

Prop.

$\forall A$ self-adjoint $\forall \varepsilon > 0$ $\exists B$ self-adjoint, $B \in \mathcal{K}(\mathcal{H})$ and $\|B\| \leq \varepsilon$ s.t.

$A+B$ has only point spectrum

$(\Leftrightarrow \mathcal{H}_{ac}(A+B) = \{0\} = \mathcal{H}_{sc}(A+B))$

$\sigma(A)$ a.c.
$\sigma_p(A+B)$

$\sigma(A) = \sigma(A+B) = \overline{\sigma_p(A+B)}$

Weyl criterion

$\lambda \in \sigma(A)$ iff

$\exists \{f_n\}_{n \in \mathbb{N}} \subset D(A)$, $\|f_n\| = 1$ and $\operatorname*{s-lim}_{n\to\infty} (A - \lambda) f_n = 0$

(A family of approximate eigenfunctions)

About $\mathcal{H}_{ac}(A)$ and $\mathcal{H}_{sc}(A)$:

Let A be self-adjoint and consider

$U_t = e^{-itA}$

Lemma :

1) If $f \in \mathcal{H}_{ac}(A)$ then $\lim_{t \to \pm\infty} \langle g, U_t f \rangle = 0 \quad \forall g \in \mathcal{H}$. It spreads on the entire $\mathcal{H}$.

This is not true in general for $f \in \mathcal{H}_{sc}(A)$ but

2) If $f \in \mathcal{H}_{sc}(A)$ then $\lim_{T \to \infty} \underbrace{\frac{1}{T} \int_0^{\pm T}}_{\text{average on time}} |\langle g, U_t f \rangle|^2 \, dt = 0 \quad \forall g \in \mathcal{H}$.

One can show that the system comes back to an inertial g less and less often, but still an infinite number of times.

Remark :

If f satisfies $Af = \mu f$, then

$\langle g, e^{-itA} f \rangle = \langle g, e^{-it\mu} f \rangle = e^{-it\mu} \langle g, f \rangle$ not convergent

Spectral representation of a self-adjoint op. A in a Hilbert space $\mathcal{H}$:

There exists a Hilbert space

$$\mathcal{K} = \int_{\sigma(A)}^{\oplus} \mathcal{K}(\lambda) \underbrace{\mu(d\lambda)}_{\text{measure on } \mathbb{R}}$$

and a unitary transformation $\mathcal{U} : \mathcal{H} \longrightarrow \mathcal{K}$ s.t.

$$\mathcal{U} A \mathcal{U}^* = \int_{\sigma(A)}^{\oplus} \lambda \, \mu(d\lambda)$$

operator of multiplication by the variable λ

This is the diagonalization of any self-adjoint op.