We consider for $\varphi \in C([a,b], \mathbb{C})$ the operator
$\int_a^b \varphi(\lambda) E(d\lambda) \equiv \int_a^b \varphi(\lambda) dE(\lambda)$ by some authors
by a limit of Riemann sums:
$$\int_a^b \varphi(\lambda) E(d\lambda) := \underset{\text{finer partition}}{\text{s-lim}} \sum_{j=1}^{n} \varphi(y_j) E((x_{j-1}, x_j])$$ notations similar to Riemannian integral

where the limit exists. For $\varphi \in C([a,b])$ the limit exists.
$\int_a^b \varphi(\lambda) E(d\lambda) \in \mathcal{B}(\mathcal{H})$.

⚠ $E(\{y\})$ can be different from 0.

Prop. (spectral integral)
Let $\{E_\lambda\}$ be a spectral family, and let $\varphi \in C([a,b], \mathbb{C})$.

1) $\left\| \int_a^b \varphi(\lambda) E(d\lambda) \right\| = \sup_{\varphi \in [a,b] \cap \operatorname{supp}(E_\lambda)} |\varphi(\lambda)|$

2) $\left[\int_a^b \varphi(\lambda) E(d\lambda)\right]^* = \int_a^b \overline{\varphi(\lambda)} E(d\lambda)$

3) $\forall f \in \mathcal{H}$, $\left\| \int_a^b \varphi(\lambda) E(d\lambda) f \right\|^2 = \int_a^b |\varphi(\lambda)|^2 m_f(d\lambda)$

4) If $\psi \in C([a,b], \mathbb{C})$, $\left(\int_a^b \varphi(\lambda) E(d\lambda)\right)\left(\int_a^b \psi(\lambda) E(d\lambda)\right) = \int_a^b \varphi(\lambda) \psi(\lambda) E(d\lambda)$.

Hints for 3):
$$\left\langle \int_a^b \varphi(\lambda) E(d\lambda) f, \int_a^b \varphi(\lambda) E(d\lambda) f \right\rangle \overset{②}{=} \left\langle f, \int_a^b \overline{\varphi(\lambda)} E(d\lambda) \int_a^b \varphi(\lambda) E(d\lambda) f \right\rangle$$
$$\overset{④}{=} \left\langle f, \int_a^b |\varphi(\lambda)|^2 E(d\lambda) f \right\rangle$$
$$= \int_a^b |\varphi(\lambda)|^2 \underbrace{\langle E(d\lambda) f, f \rangle}_{= m_f(d\lambda)}.$$

How can one generate this to $\int_{\mathbb{R}} \varphi(\lambda) E(d\lambda)$?
Easy if $\|\varphi\|_\infty < \infty$ or if $\operatorname{supp}(E_\lambda)$ is bounded (Prop. 1).
But otherwise: by ③ it is natural to define
$$D_\varphi := \left\{ f \in \mathcal{H} \;\middle|\; \int_{-\infty}^{\infty} |\varphi(x)|^2 m_f(d\lambda) < \infty \right\}$$
and we consider the unbounded operator
$\left(\int_{-\infty}^{\infty} \varphi(\lambda) E(d\lambda), D_\varphi\right)$ $\int_{-\infty}^{\infty} \varphi(\lambda) E(d\lambda) f$ is well-defined for $f \in D_\varphi$ by ③.

Question

Is D_φ dense in $\mathcal{H}$?

Yes since $\{E((-M, M))f \mid f \in \mathcal{H}\ \forall M \in \mathbb{R}_+\}$

is dense in $\mathcal{H}$ and is included in D_φ.

↳ since $\operatorname{s-lim}_{M\to\infty} E((-M, M)) = 1$

Thus

$(\int_a^b \varphi(\lambda) E(d\lambda), D_\varphi)$ is a densly defined operator in $\mathcal{H}$.

Lemma:

can be weakened

$(\int_a^b \varphi(\lambda) E(d\lambda), D_\varphi)$ is self-adjoint if $\varphi \in C(\mathbb{R}, \mathbb{R})$.

It is called the self-adjoint operator associated with $\{E_\lambda\}$ and φ.

Note that we can consider the special case

$\varphi(\lambda) = \lambda = id(\lambda)$ identity function

Thm. (Spectral thm)

There exists a bijective correspondence

between spectral families (or spectral measures) and self-adjoint operators.

The correspondense is

$A = \int_{\mathbb{R}} \lambda E(d\lambda)$, $D(A) = D_{id}$.

We often write E^A for the spectral family associated with A.

It is also natural to write

$\varphi(A) := \int_{\mathbb{R}} \varphi(\lambda) E^A(d\lambda)$

Remark: In good situation, $\varphi(A)$ can be constructed

by the Taylor expansion of φ, namely

if $\varphi(x) = \sum_{j=0}^{\infty} c_j x^j$, then $\varphi(A) = \sum_{j=0}^{\infty} c_j A^j$ whenever convergent in the strong topology.

⚠ If A is unbounded, $\sum_{j=0}^{\infty} c_j A^j$ is defined only on $\bigcap_{j=0}^{\infty} D(A^j)$,

which can be quite small.

Exercises:

- $\operatorname{supp}(E_\lambda^A) = \sigma(A)$
- $\|(A-z)^{-1}\| = \operatorname{dist}(z, \sigma(A))^{-1}$ distance on the complex plane

Let $t \in \mathbb{R}$ and consider the function $\varphi_r : \mathbb{R} \ni \lambda \longmapsto e^{-it\lambda}$, which is a bounded function. Then

$\varphi_r(A) = e^{-itA} \in B(\mathcal{H})$, and it is even a unitary operator.

In addition

$$e^{-itA} e^{-isA} = (e^{-it\cdot} e^{-is\cdot})(A) = (e^{-i(t+s)\cdot})(A) = e^{-i(t+s)A} \qquad \forall s,t \in \mathbb{R}$$

which means that

$\{e^{-itA}\}_{t\in\mathbb{R}}$ is a representation of the group $(\mathbb{R},+)$ by unitary operators

Moreover, it is strongly continuous. It means

$$\circledast := \| e^{-i(t+\varepsilon)A} f - e^{-itA} f \| \xrightarrow[\text{strongly}]{\varepsilon\to 0} 0$$

Indeed,

$$\circledast = \| e^{-itA}(e^{-i\varepsilon A} - 1) f \| \overset{\text{unitary}}{=\!=} \| (e^{-i\varepsilon A} - 1) f \| = \int_{-\infty}^{\infty} \underbrace{|e^{-i\varepsilon\lambda} - 1|^2}_{\leq 4} m^A{}_f(d\lambda)$$

$$\leq 4 \int_{-\infty}^{\infty} m^A{}_f(d\lambda) = 4\|f\|^2$$

By applying Lebesgue dominated convergence thm, one has

$$\lim_{\varepsilon\to 0} \circledast = \lim_{\varepsilon\to 0} \int_{\mathbb{R}} |e^{-i\varepsilon\lambda} - 1|^2 m^A{}_f(d\lambda) = \int_{\mathbb{R}} \lim_{\varepsilon\to 0} |e^{-i\varepsilon\lambda} - 1|^2 m^A{}_f(d\lambda)$$

$$= \int_{\mathbb{R}} 0 \, m^A{}_f(d\lambda) = 0$$

It means

$\{e^{-itA}\}_{t\in\mathbb{R}}$ is a strongly continuous unitary representation of $\mathbb{R}$.

Thm. (Stone's thm)

There is a bijective correspondence between self-adjoint operators and strongly continuous unitary representation of $\mathbb{R}$.

In particular, if $\{U_t\}_{t\in\mathbb{R}}$ is a strongly continuous unit. rep. of $\mathbb{R}$, we set

$$D(A) = \{ f \in \mathcal{H} \mid \exists \operatorname*{s-lim}_{t\to 0} \tfrac{1}{t}(U_t - 1) f \}$$

and

$$Af = \lim_{t\to 0} \frac{i}{t}(U_t - 1) f$$

In summary: bijective correspondance

spectral families $\Leftrightarrow$ spectral measures $\Leftrightarrow$ self-adjoint operators

$\Leftrightarrow$ strongly continuous unitary groups.

Remark: For suitable φ, the operator $\varphi(A)$ can also be defined by $\{e^{-itA}\}_{t\in\mathbb{R}}$:

$$\varphi(A) = \frac{1}{\sqrt{2\pi}} \int_{-\infty}^{\infty} \check{\varphi}(t) e^{-itA} dt$$

Indeed

$$\langle f, \varphi(A) f \rangle = \int \varphi(\lambda) m_f(d\lambda) = \int m_f(d\lambda) \frac{1}{\sqrt{2\pi}} \int_{\mathbb{R}} e^{-it\lambda} \check{\varphi}(t) dt$$

$$\overset{\text{Fubini}}{=\!=} \frac{1}{\sqrt{2\pi}} \int_{\mathbb{R}} \check{\varphi}(t) \left[\int e^{-it\lambda} m_f(d\lambda) \right] dt$$

$$= \langle f, \frac{1}{\sqrt{2\pi}} \int \check{\varphi}(\lambda) e^{-itA} dt \, f \rangle$$

Then, use polarization identity for

$$\langle f, \varphi(A) g \rangle = \langle f, \frac{1}{\sqrt{2\pi}} \int \check{\varphi}(t) e^{-itA} dt \, g \rangle$$