For $f \in L^2(\mathbb{R}^n)$ observe that

$$\varphi(D)f = \mathcal{F}^*\varphi(X)\mathcal{F}f = \mathcal{F}^*\big(\varphi(X)\underset{\substack{\text{Fourier}\\\text{transfm.}}}{\hat{f}}\big) = \underset{\substack{\text{Inverse Fourier}\\\text{transfm.}}}{\check{\varphi}} * f\,(x)$$

$$= \frac{1}{(2\pi)^{n/2}} \int_{\mathbb{R}^n} \check{\varphi}(y)\, f(x-y)\, dy =: \text{convolution}$$

Exercise:

In $L^2(\mathbb{R}^n)$, recall $D_j := -i\partial_j$ and $[X_k f](x) := x_k f(x)$. Then prove:

$[X_j, X_k] = 0$; $[D_j, D_k] = 0$; $[iD_j, X_k] = \delta_{jk}$ on $\mathcal{S}(\mathbb{R}^n)$

Harmonic oscillator in $L^2(\mathbb{R}^d)$

$$H := \underset{-(\partial_1^2+\dots+\partial_n^2)}{\Delta} + \omega \underset{(X_1^2+\dots+X_n^2)}{X^2} \quad \text{with } \omega \in \mathbb{R}$$

Exercise: its spectrum

This operator has to be defined on a suitable domain, and then it is self-adjoint.

Schrödinger operator in $L^2(\mathbb{R}^n)$

Consider $h: \mathbb{R}^n \longrightarrow \mathbb{R}$ as e.g.

- $h(\xi) = \xi^2$
- $h(\xi) = |\xi|$ Euclidean norm
- $h(\xi) = \sqrt{|\xi|^2+1} - 1$

The corresponding operators $h(D)$ are

free Laplace operator Δ

free relativistic Schödinger op.

free relativistic Schödinger op. of mass 1

Then we usually consider

$$H := h(D) + V(X) \quad V(x) \text{ called the potential} \quad \text{with } V: \mathbb{R}^n \longrightarrow \mathbb{R}$$

⚠ if V is not bounded we need a suitable domain for H.

Hydrogen atom in $L^2(\mathbb{R}^3)$

$$H := \Delta - \gamma \frac{1}{|X|} \quad \text{multiplication op. by an unbounded function} \quad \gamma \in \mathbb{R}$$

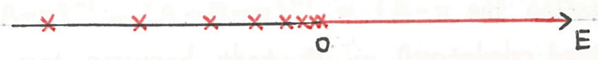
with domain $D(H) = D(-\Delta)$ is self-adjoint.

In fact $\frac{1}{|X|}$ is Δ-bounded with relative bound 0.

$\Rightarrow D(H) = D(\Delta)$ for any coupling constant γ.

One has

$$\sigma(H) = \mathbb{R}_+ \sqcup \left\{ -\left(\frac{\gamma}{2(n+1)}\right)^2 \,\middle|\, n \in \mathbb{N} \right\} \quad \text{if } \gamma > 0$$



Weyl calculus

What about $h: \mathbb{R} \times \mathbb{R} \longrightarrow \mathbb{C}$ and $h(X, D)$?

How can we define $h(X, D)$?

More than one solutions, which are not equivalent.

One of them is Weyl calculus.

We would like to define a product $\circ$ and an involution $^{\circ}$ such that

$$h_1(X,D)\, h_2(X,D) \ \text{product of two operators} = (h_1 \circ h_2)(X,D) \ \text{product of two functions}$$

and

$$h(X,D)^* \ \text{adjoint operator} = h^{\circ}(X,D) \ \text{involution of a function}$$

For $h : \mathbb{R}^n \times \mathbb{R}^n \longmapsto \mathbb{C}$ and $u \in L^2(\mathbb{R}^n)$, one sets

$$[h(X,D)u](x) := \frac{1}{(2\pi)^n} \int_{\mathbb{R}^n}\int_{\mathbb{R}^n} e^{i(x-y)\cdot\xi}\, h\left(\frac{x+y}{2}, \xi\right) u(y)\, dy\, d\xi \quad \text{called a quantization}$$

scalar product in $\mathbb{R}^n$

Exercise: if h depends only on the first n variables (or the last n variables)

one gets $h(X)$ or $h(D)$.

Also, if h is real-valued then $h(X,D)$ is symmetric

(and self-adjoint if we are lucky!)

And we have

$$\begin{cases} h_1(X)\, h_2(X) = (h_1 h_2)(X) \\ h_1(D)\, h_2(D) = (h_1 h_2)(D) \end{cases}$$

Note that there is a form for $h_1 \circ h_2$,

but it is not the pointwise multiplications,

it corresponds to a pointwise product in the first n variables,

and a convolution product in the last n variables.

And

$$h^{\circ} = \overline{h} \quad \text{complex conjugation of a function}$$