Lemma: Let $A \in \mathcal{B}(\mathcal{H})$. Then (proof as exercise)

$\sigma(A) \subset B_{\|A\|}(0) \subset \mathbb{C}$ $\Leftrightarrow$ If $z \notin B_{\|A\|}(0)$ then $z \in \rho(A)$ easier to prove

↳ ball centered at 0 with radius $\|A\|$ (using Neumann series)

Prop. Let $(A, D(A))$ be self-adjoint. Then (très important)

1) $\sigma_p(A) \subset \mathbb{R}$
2) $\sigma(A) \subset \mathbb{R}$
3) If $\lambda, \mu$ are eigenvalues of $A$, with $\mu \neq \lambda$ and if $f_\lambda, f_\mu$ are the corresponding eigenfunctions, then $f_\lambda$ and $f_\mu$ are orthogonal.

Proof:

1) Let $0 \neq f \in D(A)$ and $z \in \mathbb{C}$ s.t. $Af = zf$. Then

$$z\|f\|^2 = z\langle f, f\rangle = \langle f, zf\rangle = \langle f, Af\rangle \underset{\text{adjoint}}{\overset{\text{self-}}{=}} \langle Af, f\rangle = \langle zf, f\rangle = \bar{z}\langle f, f\rangle$$
$$= \bar{z}\|f\|^2$$

$\Rightarrow z = \bar{z} \Rightarrow z \in \mathbb{R}$.

2) Set $z = \lambda + i\varepsilon$ with $\lambda \in \mathbb{R}$, $\varepsilon \in \mathbb{R}$, $\varepsilon \neq 0$ and let us show that $z \in \underset{\text{resolvent set}}{\rho(A)}$.
(it then implies that $\sigma(A) \in \mathbb{R}$)

- $\forall f \in D(A)$ one has

$$\|(A-z)f\|^2 = \|(A-\lambda)f - i\varepsilon f\|^2 = \langle (A-\lambda)f - i\varepsilon f, (A-\lambda)f - i\varepsilon f\rangle$$
$$= \underbrace{\|(A-\lambda)f\|^2}_{>0} + \varepsilon^2\|f\|^2$$

$$\Rightarrow \|(A-z)f\| \geqslant |\varepsilon|\|f\| \begin{cases} >0, & \text{for } f \neq 0 \\ =0, & \text{for } f = 0 \end{cases}$$

$\Rightarrow A - z$ is injective $\Rightarrow A - z$ is invertible on $\mathrm{Ran}(A-z)$

- $\forall g \in \mathrm{Ran}(A-z)$, one has by the previous point

$$\|g\| = \|(A-z)\underbrace{(A-z)^{-1}g}_{\in D(A-z) = D(A)}\| \geqslant |\varepsilon|\|(A-z)^{-1}g\|$$

$\Leftrightarrow \|(A-z)^{-1}g\| \leqslant \frac{1}{|\varepsilon|}\|g\|$ $\Rightarrow (A-z)^{-1}$ is bounded on $\mathrm{Ran}(A-z)$

- Let us show that $\mathrm{Ran}(A-z)$ is dense in $\mathcal{H}$. Indeed

$$\mathrm{Ran}(A-z)^\perp = \mathrm{Ker}((A-z)^*) = \mathrm{Ker}(A^* - \bar{z}) \underset{\text{adjoint}}{\overset{\text{self-}}{=}} \mathrm{Ker}(A - \bar{z})$$
$= \{0\}$ since all eigenvalues are real $\Rightarrow \mathrm{Ran}(A-z)$ is dense in $\mathcal{H}$

$\Rightarrow (A-z)^{-1}$ is a densely defined bounded linear operator in $\mathcal{H}$.
By the first Remark in p. 29, $(A-z)^{-1}$ admits a continuous bounded extension on $\mathcal{H}$
$\Leftrightarrow (A-z)^{-1} \in \mathcal{B}(\mathcal{H})$ $\Leftrightarrow z \in \rho(A)$.

3) One has (for $\lambda, \mu \in \sigma_p(A) \subset \mathbb{R}$, $\lambda \neq \mu$)

$$\lambda \langle f_\lambda, f_\mu \rangle = \langle \lambda f_\lambda, f_\mu \rangle = \langle A f_\lambda, f_\mu \rangle \overset{\text{self-}}{\underset{\text{adjoint}}{=}} \langle f_\lambda, A f_\mu \rangle = \langle f_\lambda, \mu f_\mu \rangle$$
$$= \mu \langle f_\lambda, f_\mu \rangle \Rightarrow \langle f_\lambda, f_\mu \rangle = 0 \qquad \square$$

## II.3 Perturbation theory

Let $(A, D(A))$ be a self-adjoint op., and consider $(B, D(B))$ symmetric op.,

Question: when is $A+B$ self-adjoint?

1) easy case: if $B = B^* \in \mathcal{B}(\mathcal{H})$, then $(A+B, D(A))$ is self-adjoint (exercise)

What about $B \notin \mathcal{B}(\mathcal{H})$?

Def. In the above framework, $B$ is A-bounded (or bounded with respect to A) if $D(A) \subset D(B)$ and $\exists\, \alpha, \beta \geq 0$:

$$\|Bf\| \leq \alpha \|Af\| + \beta \|f\| \qquad \forall f \in D(A)$$

The infimum of $\alpha$ is called the A-bound of B.

Remark:

- If $B \in \mathcal{B}(\mathcal{H})$, the A-bound is 0.
- One can have an A-bound equal to 0 even if $B \notin \mathcal{B}(\mathcal{H})$ $\quad \alpha_n \xrightarrow{n\to\infty} 0$, $\beta_n \longrightarrow \infty$

Thm. (Kato-Rellich thm)

Let $(A, D(A))$ be self-adjoint, and $(B, D(B))$ be symmetric and A-bounded with A-bound $< 1$. Then

1) $(A+B, D(A))$ is self-adjoint;
2) $B$ is also $(A+B)$-bounded.

Remark: The statement is not correct if the A-bound $= 1$.

Take $B = -A$ but then $(\mathbf{0}, D(A))$ is not self-adjoint. the domain of $\mathbf{0}^*$ is not $D(A)$

Thm. (The second resolvent equation)

In the setting of the previous thm, one has for $z \in \rho(A) \cap \rho(A+B)$:

$$(A-z)^{-1} - (A+B-z)^{-1} = (A-z)^{-1} B (A+B-z)^{-1} = (A+B-z)^{-1} B (A-z)^{-1}$$

It is not assumed that $B$ is bounded, but for $B(A+z)^{-1}$

$\mathcal{H} \xrightarrow{(A-z)^{-1}} D(A) \subset D(B) \xrightarrow{B}$ meaningful;

same for $B(A+B-z)^{-1}$ because of Kato-Rellich thm.

Formal proof:

$$(A-z)^{-1} - (A+B-z)^{-1} = (A-z)^{-1} \underline{(A+B-z)(A+B-z)^{-1}} - \underline{(A-z)^{-1}(A-z)}(A+B-z)^{-1}$$
$$= (A-z)^{-1} [(A+B-z) - (A-z)] (A+B-z)^{-1}$$
$$= (A-z)^{-1} B (A+B-z)^{-1}$$

## II.4 Examples

Let $\mathcal{H} = L^2(\mathbb{R}^n) \ni f$.

If $\varphi : \mathbb{R}^n \longmapsto \mathbb{C}$ we set $M_\varphi \equiv \varphi(X) \equiv \varphi(Q)$ acting as $(M_\varphi f)(x) = \varphi(x) f(x)$

(multiplication operators)

One has $D(M_\varphi) = \{ f \in \mathcal{H} \mid M_\varphi f \in \mathcal{H} \}$

Examples

$X_j$, or $X_j X_k$, with $j, k \in \{1, 2, \cdots, n\}$

More generally $X^\alpha = X_1^{\alpha_1} X_2^{\alpha_2} \cdots X_n^{\alpha_n}$ with $\alpha \in \mathbb{N}^n$

E.g. $X^2 := X_1^2 + X_2^2 + \cdots + X_n^2$

Fourier transform (in Schwarz space)

For $f \in \mathcal{S}(\mathbb{R}^n)$ we set

$$[\mathcal{F}^{-1} f](\xi) = \frac{1}{(2\pi)^{n/2}} \int_{\mathbb{R}^n} e^{\pm i \xi \cdot x} f(x)\, dx$$

$\mathcal{F}$ is a unitary operator : $\mathcal{F}^* = \mathcal{F}^{-1}$

- For the multiplication operators

$$\begin{aligned}
(2\pi)^{n/2} [X_j \mathcal{F} f](\xi) &= (2\pi)^{n/2} \xi_j [\mathcal{F} f](\xi) \\
&= \xi_j \int_{\mathbb{R}^n} e^{-i\xi\cdot x} f(x)\, dx = \int_{\mathbb{R}^n} i(\partial_{x_j} e^{-i\xi\cdot x}) f(x)\, dx \\
\text{(integate by part)} \quad &= \int_{\mathbb{R}^n} e^{-i\xi\cdot x} [-i\partial_j f](x)\, dx \\
&= (2\pi)^{n/2} [\mathcal{F}(-i\partial_j f)](\xi)
\end{aligned}$$

$\Rightarrow X_j \mathcal{F} = \mathcal{F}(-i\partial_j)$

$\Leftrightarrow \mathcal{F}^* X_j \mathcal{F} = -i\partial_j =: P_j \equiv D_j$ (differential operator)

In particular we consider

$\Delta := D_1^2 + D_2^2 + \cdots + D_n^2$ (Laplacian)

one gets

$\mathcal{F} \Delta \mathcal{F}^* = X^2 = X_1^2 + X_2^2 + \cdots + X_n^2$

Hence since $\mathcal{F}$ is unitary ($\Rightarrow$ preserving the norm),

$\sigma(\Delta) = \sigma(X^2) = \mathrm{Ran}(x \mapsto x^2) = [0, \infty)$

For $\varphi : \mathbb{R}^n \longmapsto \mathbb{R}$, we can also define :

$\varphi(D) := \mathcal{F}^* \underset{M_\varphi}{\underset{\shortparallel}{\varphi(X)}} \mathcal{F}$ (convolution operator)