Chapter II : Unbounded and self-adjoint operators

Reminder: Special classes in $B(\mathcal{H})$

- Self-adjoint op. — Hamiltonian quantum system
- Projection — Energy region
- unitary op. — Evolution
- isometries — Non-reversible systems
- partial isometries — Scattering theory
- Finite rank op. — Easy perturbation theory
- compact op. — More advanced perturbation theory

II.1 : Some defination

Def. A linear operator on $\mathcal{H}$ is a pair $(A, D(A))$
with $D(A) \subset \mathcal{H}$ a linear manifold,
and $A : D(A) \longmapsto \mathcal{H}$ is a linear map. $A(f+\lambda g) = Af + \lambda Ag$
We say that $(A, D(A))$ is densely defined if $D(A)$ is dense in $\mathcal{H}$.

$$\mathrm{Ran}(A) = \{f \in \mathcal{H} \mid \exists g \in D(A) : f = Ag\} = AD(A)$$

$$\mathrm{Ker}(A) = \{f \in D(A) \mid Af = 0\}$$

Clearly, if A, B are linear operators then
- $A+B$ is only defined on $D(A) \cap D(B)$
- AB is only defined on $\{f \in D(B) \mid Bf \in D(A)\}$
- $[A,B] = AB - BA$ only defined on a suitable domain.

Example:

$\mathcal{H} = L^2(\mathbb{R}) \ni f$.
Let $X \, (= Q = M_{id})$ be the operator given by $[Xf](x) = xf(x)$
Some domains are $C_c^\infty(\mathbb{R})$, $\mathcal{S}(\mathbb{R})$, or
$\{f \in L^2(\mathbb{R}) \mid \int_{\mathbb{R}} |xf(x)|^2 dx < \infty\} =: \mathcal{H}_1(\mathbb{R})$: weighted Hilbert space.
Let us check that $X \notin B(\mathcal{H})$.
Take $y \in \mathbb{R}$ and consider $f_y : \mathbb{R} \longmapsto \mathbb{R}$

$$f_y(x) = \begin{cases} 1 & \text{if } x \in [y, y+1] \\ 0 & \text{otherwise} \end{cases} \Rightarrow \|f_y\| = 1$$

$$\Rightarrow \|Xf_y\|^2 = \int_{\mathbb{R}} |xf_y(x)|^2 dx = \int_y^{y+1} x^2 dx = \frac{1}{3}x^3 \Big|_y^{y+1} = y^2 + y + \frac{1}{3}$$

If $X \in B(\mathcal{H})$ then

$$\|X\| = \sup_{f \in \mathcal{H}, \|f\| = 1} \|Xf\| = \infty \quad \text{by taking } f = f_y \text{ with } y \to \infty$$

$\Rightarrow X \notin B(\mathcal{H})$

Remark: $C_c^\infty(\mathbb{R})$, $\mathcal{S}(\mathbb{R})$, $\mathcal{H}_1(\mathbb{R})$ are dense in $\mathcal{H}$.

Def. Let $(A, D(A))$ and $(B, D(B))$ be linear operators in $\mathcal{H}$.
If $D(A) \subset D(B)$ and $Af = Bf \ \forall f \in D(A)$ then we say that $(B, D(B))$ is an extension of $(A, D(A))$, or $(A, D(A))$ is an restriction of $(B, D(B))$ to $D(A)$.

For example

$(X, C_c^\infty(\mathbb{R})) \subset (X, S(\mathbb{R})) \subset (X, \mathcal{H}_1(\mathbb{R}))$.

Remark: If $(A, D(A))$ is a densely defined linear operator and if $\exists\, c > 0$ with $\|Af\| \leq c\|f\| \ \forall f \in D(A)$ then A can be continuously extended to an element of $B(\mathcal{H})$.

$\rightsquigarrow$ closed operators: natural generalization of bounded operators.

If such a c does not exist, then we say that $(A, D(A))$ is a densely defined unbounded operators.

Def. Let $(A, D(A))$ be densely defined. Set $\longrightarrow$ implying that A^*f is unique

$D(A^*) := \{ f \in \mathcal{H} \mid \exists! f^* \in \mathcal{H} \text{ with } \langle f^*, g \rangle = \langle f, Ag \rangle \ \forall g \in D(A) \}$

$A^* f := f^* \quad \forall f \in D(A^*)$.

Clearly $\langle A^* f, g \rangle = \langle f, Ag \rangle \ \forall g \in D(A),\ f \in D(A^*)$ $D(A^*)$ is dense if A is closable

Exercise:

- $\mathrm{Ker}(A^*) = \mathrm{Ran}(A)^\perp$ 2~3 lines
- If $(A, D(A)) \subset (B, D(B))$ with $D(A)$ dense, then $(B^*, D(B^*)) \subset (A^*, D(A^*))$ be careful about the def of A^*, B^*

Def. $(A, D(A))$ densly defined is self-adjoint if

$D(A^*) = D(A)$ and $A^* f = Af \ \forall f \in D(A)$

Remark: For any $(A, D(A))$ self-adjoint,

$\langle Af, g \rangle = \langle f, Ag \rangle \ \forall f, g \in D(A)$ (*)

⚠ But this equality does not imply that A is self-adjoint.

Def. A densely operator $(A, D(A))$ satisfying (*) is called symmetric.

$\{\text{symmetric op.}\} \supset \{\text{self-adjoint op.}\}$

Def. A symmetric operator $(A, D(A))$ is essentially self-adjoint if it has a unique self-adjoint extension.
In this case, we call $D(A)$ a core for A.

II.2 Resolvent and spectrum

Let $(A, D(A))$ be a densely defined linear operator.

A is invertible if $\mathrm{Ker}(A) = \{0\}$, and then $A^{-1} : R(A) \longrightarrow D(A)$ is bijectiv ← range

If $R(A) = \mathcal{H}$, then $A^{-1} \in \mathcal{B}(\mathcal{H})$.

Def. Let $(A, D(A))$ be self-adjoint. This theory is valid also for closed operators.

We set the resolvent set

$$\rho(A) = \{ z \in \mathbb{C} \mid (A-z)^{-1} \in \mathcal{B}(\mathcal{H}) \} \subset \mathbb{C}$$
$$= \{ z \in \mathbb{C} \mid \mathrm{Ker}(A-z) = \{0\} \text{ and } \mathrm{Ran}(A-z) = \mathcal{H} \} \subset \mathbb{C}$$

and the spectrum of A

$$\sigma(A) = \mathbb{C} \setminus \rho(A)$$ For Hermitian matrices this is the set of eigenvalues

Example:

Consider $\mathcal{H} = L^2(\mathbb{R}) \ni f$,

$\varphi \in L^\infty(\mathbb{R}, \mathbb{R})$, $\varphi(X) \equiv M_\varphi \equiv \varphi(Q)$

↖ real-valued

$[\varphi(X) f](x) = \varphi(x) f(x)$

One has $\sigma(\varphi(X)) = \overline{\mathrm{Ran}(\varphi)} = \overline{\varphi(\mathbb{R})}$ ← closure in $\mathbb{R}$

Indeed $\varphi(X) - z$ is boundedly invertible if

$\varphi(x) - z \neq 0 \quad \forall x \in \mathbb{R}$

Properties:

- $\rho(A)$ is an open set in $\mathbb{C}$ $\Leftrightarrow$ $\sigma(A)$ is closed in $\mathbb{C}$.
- (First resolvent equation) if $z_1, z_2 \in \rho(A)$ then
$$(A-z_1)^{-1} - (A-z_2)^{-1} = (z_1 - z_2)(A-z_1)^{-1}(A-z_2)^{-1}$$ one-line proof
- $(A-z_1)^{-1}$ and $(A-z_2)^{-1}$ commute.

Def. Let $z \in \mathbb{C}$ and suppose that

$$\exists f \in D(A),\ f \neq 0 : \ Af = zf$$

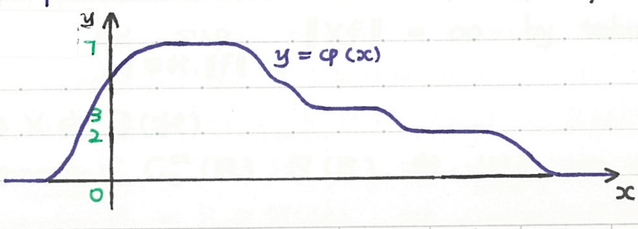
Then z is called an eigenvalue of A, with the set of all eigenvalues $\sigma_p(A)$, and f an associated eigenfunction or eigenvector.

Remark:

In the previous example with $\varphi = \tanh$, then

$\sigma(\tanh(X)) = [-1, 1]$ but $\sigma_p(\tanh(X)) = \phi$.

If φ takes a value c on a set of measure $\neq 0$, then $c \in \sigma_p(\varphi(X))$.



$\sigma(\varphi(X)) = [0, 7]$

$\sigma_p(\varphi(X)) = \{0, 2, 3, 7\}$