Introduction to Functional Analysis

Distribution theory

1) Test functions and distributions

Aim: to understand

$$\int f(x)\,\delta_0(x)\,dx = f(0)$$

Who is $\delta_0 \equiv \delta$?

Answer: δ is a linear and continuous functional on smooth functions with compact support.

Reminder: Consider $f : \mathbb{R}^n \longmapsto \mathbb{R}$ (or $\mathbb{C}$) and

$$[\partial_j f](x) \equiv [D_j f](x) \equiv [\nabla_j f](x) = \lim_{\varepsilon \to 0} \frac{f(x+\varepsilon E_j) - f(x)}{\varepsilon}$$

for $x \in \mathbb{R}^n$, $E_j = \begin{pmatrix} 0 \\ 0 \\ 1 \\ \vdots \\ 0 \end{pmatrix} \leftarrow$ position j and $j \in \{1, \cdots, n\}$

For $m \in \mathbb{N}$, we set

$$\partial_j^m f := \underbrace{\partial_j \partial_j \cdots \partial_j}_{m \text{ times}} f$$

If $\alpha \in \mathbb{N}^n = \{\alpha = (\alpha_1, \alpha_2, \cdots, \alpha_n)\}$ (each $\alpha_i \in \mathbb{N}$) multi-index then

$$\partial^\alpha f := \partial_1^{\alpha_1} \partial_2^{\alpha_2} \cdots \partial_n^{\alpha_n} f$$

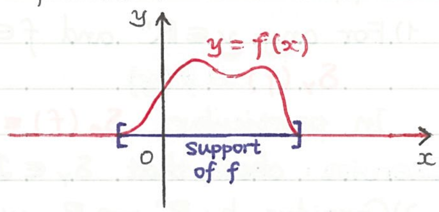
Def. A function $f : \mathbb{R}^n \longmapsto \mathbb{C}$ is **smooth** if $\partial^\alpha f$ exists and is a continuous function for any $\alpha \in \mathbb{N}^n$.

Remark: 1) For such function, $\partial_j \partial_k f = \partial_k \partial_j f$ for $j, k \in \{1, \cdots, n\}$

2) Any derivative of a smooth function is a smooth function.

Def. For any $f : \mathbb{R}^n \longmapsto \mathbb{C}$ the **support of f** is

$\overline{\{x \in \mathbb{R}^n \mid f(x) \neq 0\}}$ ← closure in $\mathbb{R}^n$



Example: $n = 1$, $f(x) = x \Rightarrow$ the support of f is $\mathbb{R}$.

We write **supp(f)** for the support.

Notation: We write $C^\infty(\mathbb{R}^n)$ either $C^\infty(\mathbb{R}^n, \mathbb{R})$ or $C^\infty(\mathbb{R}^n, \mathbb{C})$ for the set of all smooth functions on $\mathbb{R}^n$.

Def. $f \in C^\infty(\mathbb{R}^n)$ is a **test function** if its support is **compact** ($\equiv$ bounded).

We write $\mathcal{D}(\mathbb{R}^n)$ for the set of all test functions.

Example: For $x = (x_1, \cdots, x_n) \in \mathbb{R}^n$, we set $\|x\| = \sqrt{x_1^2 + \cdots + x_n^2}$ and

$$f(x) = \begin{cases} \exp\left(-\frac{1}{1-\|x\|}\right) & \text{if } \|x\| \leq 1 \\ 0 & \text{if } \|x\| > 1 \end{cases}$$

This function has support in $\overline{B(\mathbf{0}, 1)}$ ($\mathbf{0} \in \mathbb{R}^n$, $1 \in \mathbb{R}$) ball of radius 1 and centred at $\mathbf{0} \in \mathbb{R}^n$ $= \overline{\{y \in \mathbb{R}^n \mid \|y\| < 1\}}$

Remark: $\mathcal{D}(\mathbb{R}^n)$ is stable for addition ($f, g \in \mathcal{D}(\mathbb{R}^n) \Rightarrow \forall \lambda \in \mathbb{R}$ or $\mathbb{C}$: $f + \lambda g \in \mathcal{D}(\mathbb{R}^n)$) and for multiplication ($fg \in \mathcal{D}(\mathbb{R}^n)$).

This is an **algebra**.

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Def. (Convergence in $\mathcal{D}(\mathbb{R}^n)$)

A sequence of function $\{f_j\}_{j\in\mathbb{N}} \subset \mathcal{D}(\mathbb{R}^n)$ converges to $f_\infty \in \mathcal{D}(\mathbb{R}^n)$ if

1) $\sup_{x\in\mathbb{R}^n} |\partial^\alpha f_j(x) - \partial^\alpha f_\infty(x)| \xrightarrow{j\to\infty} 0$ for any $\alpha \in \mathbb{N}^n$

2) $\mathrm{supp}(f_j) \subset B(0,R)$ for some large $R \in \mathbb{R}$ (independent of j) and all $j \in \mathbb{N}$.

Remark: This is stronger (and including) than uniform convergence.

Def. A distribution T on $\mathbb{R}^n$ is a continuous linear function from $\mathcal{D}(\mathbb{R}^n)$ to $\mathbb{R}$ ($\mathbb{C}$).

More explicitly, $T : \mathcal{D}(\mathbb{R}^n) \longmapsto \mathbb{C}$ or $\mathbb{R}$ such that

1) $\underbrace{T(\underbrace{f_1 + \lambda f_2}_{\in\mathcal{D}(\mathbb{R}^n)})}_{\in\mathbb{C}} = \underbrace{Tf_1}_{\in\mathbb{C}} + \underbrace{\lambda \underbrace{Tf_2}_{\in\mathbb{C}}}_{\in\mathbb{C}}$.

2) $\underbrace{T(f_j)}_{\in\mathbb{C}} \xrightarrow{j\to\infty} \underbrace{T(f_\infty)}_{\in\mathbb{C}}$ if $\{f_j\}$ converges to f_∞ in $\mathcal{D}(\mathbb{R}^n)$

The family of all distributions is denoted by $\mathcal{D}'(\mathbb{R}^n)$ dual space of $\mathcal{D}(\mathbb{R}^n)$.

Again, 2 distributions can be added:

$$(T_1 + \lambda T_2) f := T_1 f + \lambda T_2 f$$

⚠ Multiplication is not well-defined.

We often write $Tf = T(f) = \langle T, f \rangle$ duality.

Examples:

1) For any $y \in \mathbb{R}^n$ and $f \in \mathcal{D}(\mathbb{R}^n)$, we set Dirac delta distribution at y

$$\delta_y(f) := f(y)$$

In particular $\delta_0(f) \equiv \delta(f) = f(0)$

Exercise: check that $\delta_y \in \mathcal{D}(\mathbb{R}^n)$

2) Consider $h : \mathbb{R}^n \longmapsto \mathbb{R}$ with

$$h \in L^1_{loc}(\mathbb{R}^n) := \{h : \mathbb{R}^n \longmapsto \mathbb{R} \mid \int_{B(y,1)} |h(x)|\, dx < \infty \ \forall y \in \mathbb{R}^n\}$$ very big set!

and set

$$T_h(f) := \int_{\mathbb{R}^n} f(x) h(x)\, dx$$

and observe that $T_h(f)$ is a distribution.

Exercise: check →

Since for any $h \in L^1_{loc}(\mathbb{R}^n)$ we can define $T_h \in \mathcal{D}'(\mathbb{R}^n)$

we can identify $L^1_{loc}(\mathbb{R}^n)$ AS A SUBSET OF $\mathcal{D}'(\mathbb{R}^n)$.

"Any" function is a distribution by identifying h with T_h.

For that reason, we sometimes write $\int f(x)\,\delta_0(x)\,dx$ for $\delta_0(f)$.

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Examples:

3) For any $\alpha \in \mathbb{N}^n$, we set

$$\delta_y^\alpha(f) := \delta_y(\partial^\alpha f) = [\partial^\alpha f](y)$$

Exercise: check that $\delta_y^\alpha \in \mathcal{D}'(\mathbb{R}^n)$

Proposition: (your friend)

$T \in \mathcal{D}'(\mathbb{R}^n)$ if and only if

1) $T : \mathcal{D}(\mathbb{R}^n) \longmapsto \mathbb{R}$ is linear and

2) $\forall y \in \mathbb{R}^n$ and $R > 0$: $\exists c > 0$ and $m \in \mathbb{N}$ with

$$|T(f)| \leq c \sum_{\substack{|\alpha| \leq m \\ \| \\ \alpha_1+\alpha_2+\cdots+\alpha_n \leq m;\ \alpha \in \mathbb{N}^n}} \boxed{\|\partial^\alpha f\|_\infty} := \sup_{x \in \mathbb{R}^n} |\partial^\alpha f(x)| \quad \text{for all } f \in \mathcal{D}(\mathbb{R}^n) \text{ with } \operatorname{supp}(f) \subset B(y, R).$$

Remark: T is of finite order if we can choose one $\underset{\text{indep of}}{\overset{\text{order}}{m}}$ for any y and R.

Then, T_h and δ_y are of order 0

while δ_y^β is of order $|\beta| := \beta_1 + \cdots + \beta_n$.

The set of test functions is very limited so that it is difficult to find an example. So is the def of distribution $T : \mathcal{D}(\mathbb{R}^n) \mapsto \mathbb{R}$ or $\mathbb{C}$ too limited? The distribution δ_y can act on almost every function which is continuous at y. Can other distributions have larger domains too?