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Calculus I

I) Number, Limits & functions

I.1) Reminder

$\mathbb{N} = \{0, 1, 2 \dots\}$

$\mathbb{Z} = \{\dots, -2, -1, 0, 1, 2, \dots\}$

$\mathbb{Q} = \left\{\frac{m}{n},\ m, n \in \mathbb{Z},\ n \neq 0\right\}$

π, $0.999\ldots = 1$

$\mathbb{R}$ = all numbers, $\mathbb{R}^* = \mathbb{R} \setminus \{0\}$

Def: for $a, b \in \mathbb{R}$,

$(a,b) =]a,b[$

open interval $= \{x \in \mathbb{R} \text{ with } a < x < b\}$

$[a,b]$

closed interval $= \{x \in \mathbb{R} \mid a \leq x \leq b\}$

- Powers:

Let $x \in \mathbb{R}$, $x \geq 0$

$x^n = x \cdot x \cdot x \cdots x$ for $n \in \mathbb{N}$ with $x^0 := 1$

$x^{\frac{1}{n}} = y \geq 0$ with

$y^n = x$, $n \in \mathbb{N}^* := \mathbb{N} \setminus \{0\}$

$x^{\frac{n}{m}} = (x^n)^{\frac{1}{m}} = (x^{\frac{1}{m}})^n$

$x^{-\frac{n}{m}} = 1 / x^{\frac{n}{m}}$

What is $\sqrt{2}^{\pi}$?

I.2) Limit

Let X be a subset of $\mathbb{R}$

Def: $(a_n)_{n \in \mathbb{N}}$ is called a "sequence" in X if $a_i \in X$ for any i.

Question What is the limit of $(a_n)_{n \in \mathbb{N}}$ when n goes to ∞ ?

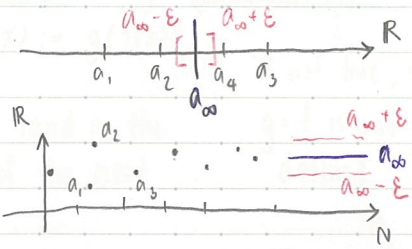
We write $\lim_{n \to \infty} a_n$ if it exists.

Ex. $a_n = \frac{1}{1+n}$, the limit

$\lim_{n \to \infty} a_n = 0$

Def A sequence $(a_n)_{n \in \mathbb{N}} \subset X$ is convergent in X.
if $\exists\, a_\infty \in X$ such that for any $\varepsilon > 0$, $\exists N \in \mathbb{N}$ with $|a_n - a_\infty| \leq \varepsilon$ for any $n \geq N$.

$a_\infty - \varepsilon \leq a_n \leq a_\infty + \varepsilon$



⚠ There are plenty of sequences which do not converge:

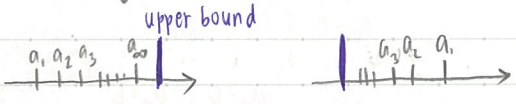
ex: • $a_n = (-1)^n$
• $a_n = n$
• $a_n = \sum_{k=1}^{n} \frac{(-1)^{k+1}}{2k-1}$ → $a_1 = 1$, $a_2 = 1 - \frac{1}{3}$, $a_3 = 1 - \frac{1}{3} + \frac{1}{5}$, $a_4 = 1 - \frac{1}{3} + \frac{1}{5} - \frac{1}{7}$

ex: $\lim_{n\to\infty} a_n = \frac{\pi}{4}$???

This sequence in $\mathbb{Q}$ does not converge in $\mathbb{Q}$, but it does converge in $\mathbb{R}$, and the limit is $\pi/4$.

Properties: Suppose $(a_n)_{n\in\mathbb{N}}$ converges to a_∞ and $(b_n)_{n\in\mathbb{N}}$ converges to b_∞. Then $(a_n + b_n)_{n\in\mathbb{N}}$ converges to $a_\infty + b_\infty$, and $(a_n b_n)_{n\in\mathbb{N}}$ converges to $a_\infty b_\infty$.

Thm Any increasing sequence which is upper bounded is convergent, and any decreasing sequence which is lower bounded is convergent. in $\mathbb{R}$!!!



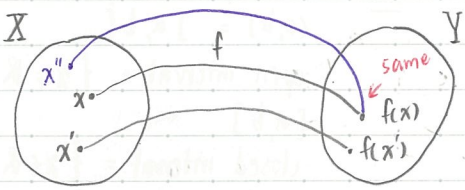
1.3) Function

Recall that a set X is a collection of object

$$X = \{x_1, x_2, \dots\}$$

Def A function (or a map) f from a set X to set Y is a rule which associates one element Y to every element of X.

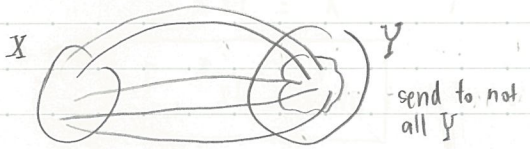
It means, to any $x \in X$ $\exists!$ $y = f(x) \in Y$ (∃! : There exists unique)



X is called the "domain" of f, denoted by $\mathrm{Dom}(f)$.

Y ——— "codomain" of f (or tangent space)

$f(X) = \mathrm{Ran}(f) = \mathrm{Im}(f)$ (range, image ↘ don't use this one)
$= \{y \in Y \mid \exists x \in X \text{ with } y = f(x)\}$
is called the range of f.

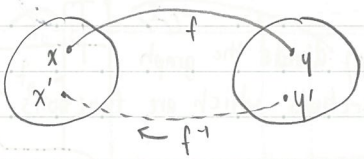


Notation: We write $X \ni x \longmapsto f(x) \in Y$
or $f: X \to Y$
$x \to f(x)$

⚠ $f(x)$ is not a function. It is an element of Y.
We can write f or $f(\cdot)$ for the function.

Def Let $f: X \to Y$

- f is injective if $\forall x_1, x_2 \in X$ but $x_1 \neq x_2$, then $f(x_1) \neq f(x_2)$
  all x maps to different y
- f is surjective if $\text{Ran}(f) = Y$
  $\Leftrightarrow \forall y \in Y, \exists x \in X$ with $f(x) = y$.
  reaches everybody
- f is bijective if f is both injective and surjective.

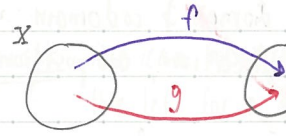


Remark Whenever $f: X \to Y$ is bijective, we can define $f^{-1}: Y \to X$ with $f^{-1}(f(x)) = x$ and $f(f^{-1}(y)) = y$.

Ex./
1) $f: \mathbb{R} \ni x \mapsto x^2 \in \mathbb{R}$
   ⇨ not inj ex. $x = +2, -2$
   ⇨ not surj ex. $y = -3$ doesn't reach
2) $f: \mathbb{R}^+ \ni x \mapsto x^2 \in \mathbb{R}$
   ⇨ inj, not surj.
3) $f: \mathbb{R} \ni x \mapsto x^2 \in \mathbb{R}^+$
   ⇨ surj, not inj
4) $f: \mathbb{R}^+ \ni x \mapsto x^2 \in \mathbb{R}^+$
   ⇨ bijective

Remark: A function is always surjective in its range.
$\text{Ran}(f) = \text{Ran}(f)$

1.4) Operation with function



define first ↑

Question: can you define $f+g$? (no)

Addition: Let X be a set, but let $Y = \mathbb{R}$
If $f, g: X \to \mathbb{R}$, then we can define $f+g$ by $(f+g)(x) = \underbrace{f(x)}_{\in \mathbb{R}} + \underbrace{g(x)}_{\in \mathbb{R}}$

not always trivial

Multiplication: We can also define fg by $(fg)(x) = f(x)\,g(x)$

Composition: If $f: X \to Y$ and $g: Y' \to Z$ with $Y' \subset Y$

Then we can define $g \circ f$ if $\text{Ran}(f) \subset Y'$
by $(g \circ f)(x) := g(f(x))$

if not true, then $g \circ f$ is not well defined.

➢ have to land in the domain of the next function.

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Reminder:

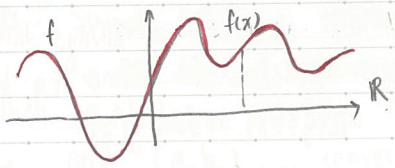
- $\mathbb{N}, \mathbb{Z}, \mathbb{Q}, \mathbb{R}$
- limits: ... for any ε, find $N(\varepsilon)$
- functions: injective, surjective, bijective
  - ⚠ depends on domain & codomain
- addition, multiplication, composition

II.) Graphs & Curves

II.1) Graphs

Let $f: \mathbb{R} \to \mathbb{R}$, and condiser the "picture".

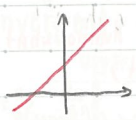
$\mathbb{R}^2$ — The plane (2-dimensional)

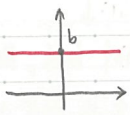


The graph of f in $\{(x,y) \in \mathbb{R}^2 \mid x \in \mathbb{R}, y = f(x)\}$

For any x, there is only 1 y.

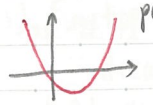
II.2) Standard examples.

1)  graph of a function $f(x) = ax + b$ (a: slope)

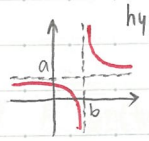
2) 

graph of $f(x) = b$.

"Not unique" (can substitute t with $2t$ and the graph is same) ↑ has an expression in t

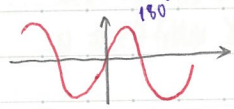
3) 

graph of the function $f(x) = ax^2 + bx + c$

4) 

graph of a function $f(x) = a + \frac{c}{x-b}$

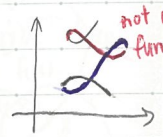
5) $f(x) = \cos(x)$, $f(-x) = f(x)$ — even function

6) 

$f(x) = \sin(x)$, $f(-x) = -f(x)$ — odd function

II.3) Curves

(1 x ↔ many y)



to study, divide the graph to small sections, which are functions

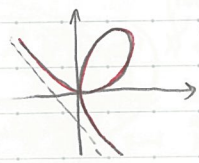
Many objects like that are not the graph of a function.

Examples:

① $\{(x,y) \in \mathbb{R}^2 \mid x^2 + y^2 = 1\}$

② $\{(x,y) \in \mathbb{R}^2 \mid \left(\frac{x}{a}\right)^2 + \left(\frac{y}{b}\right)^2 = 1$, with $a, b > 0\}$

③ $\{(x,y) \in \mathbb{R}^2 \mid x = \frac{3t}{1+t^3}, y = \frac{3t^2}{1+t^3}, t \in \mathbb{R}\}$ — parametric curve with parameter t

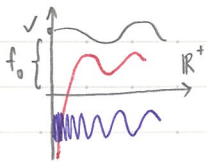


"The folium of Descartes"

These are not the graph of any func. from $\mathbb{R}$ to $\mathbb{R}$, but they are curves or parametric curves.

⚠ A curve can be locally studied as a function.

III) The Derivative



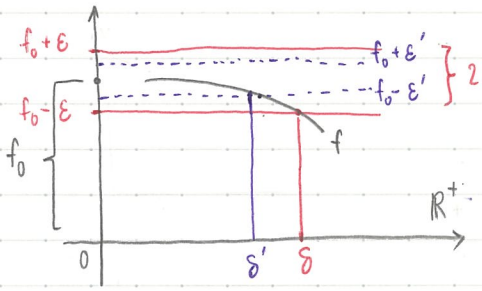
III.1) Limits and Continuity

Def: Let $f: (0,\infty) \to \mathbb{R}$

We say that f has a limit at 0 if there exists a value $f_0 \in \mathbb{R}$ such that for any $\varepsilon > 0$ (small but arbitrary), there exists $\delta > 0$ with

$$|f(x) - f_0| \leq \varepsilon \quad \text{for any } x \in (0, \delta).$$

$$f_0 - \varepsilon \leq f(x) \leq f_0 + \varepsilon$$



If f has a limit at 0, we write

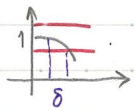
$$\lim_{x \searrow 0} f(x) = f_0$$
$$\| $$
$$\lim_{x \to 0^+} f(x) = f_0$$

[Our job is to find δ now.]

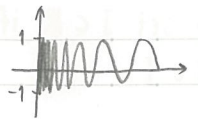
Examples

① $f: (0,\infty) \to \mathbb{R}$, $f(x) = x^n$ has a limit

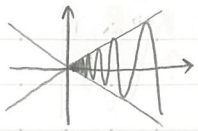
② $f: (0,\infty) \to \mathbb{R}$, $f(x) = \cos(x)$ has a limit.



③ $f(x) = x \sin\left(\frac{1}{x}\right)$



$f(x) = \sin\left(\frac{1}{x}\right)$ has no limit at 0.



$f(x) = x \sin\left(\frac{1}{x}\right)$ has the limit 0 at 0.

Remark We can also consider limit from the left for a function

$$f: (-\infty, 0) \to \mathbb{R}$$

If it has a limit, one writes

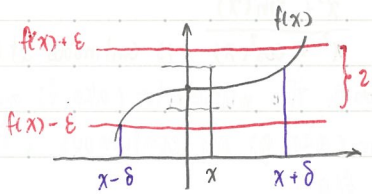
$$\lim_{x \nearrow 0} f(x) = f_0 = \lim_{x \to 0^-} f(x)$$

Def ($\forall$ = for any, $\exists$ = there exists) $f: \mathbb{R} \to \mathbb{R}$ is continuous at $x \in \mathbb{R}$ if

$(\forall)\, \varepsilon > 0$, $(\exists)\, \delta > 0$ such that

$$|f(x+h) - f(x)| \leq \varepsilon \quad \text{for any } h \in [-\delta, \delta]$$

$$f(x) - \varepsilon \leq f(x+h) \leq f(x) + \varepsilon$$



$\iff$ $\forall \varepsilon > 0$, $\exists \delta > 0$ such that

$|f(y) - f(x)| \leq \varepsilon$ for $y \in \mathbb{R}$ with $|y - x| \leq \delta$ (y is just another variable)

if and only if ↩

Remark f is continuous at x iff

$$\lim_{y \searrow x} f(y) = \lim_{y \nearrow x} f(x)$$

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f is continuous on a set $I \subset \mathbb{R}$ if f is continuous at any $x \in I$.

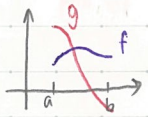
For fun: What means $f : \mathbb{R} \to \mathbb{R}$ is continuous on $\mathbb{Z}$?

(a,b), $[a,b]$, etc.

The set of all continuous function on I (interval) is denoted by $C(I)$.

These are all continuous.

Properties Consider $f, g \in C(I)$ and let $\lambda \in \mathbb{R}$



1) $\lambda f + g \in C(I)$
2) $fg \in C(I)$
3) f/g is continuous if $g(x) \neq 0 \quad \forall x \in I$

Remark How to know that

$$f(x) = \frac{x^3 + \sin(x)}{x^2 + \cos^2(x)}$$ is continuous at 0.

Show that both the numerator (aka. f) and the denominator (aka. g) are continuous. (from the properties: f/g)

④ If we consider $f(x) = \cos(x^2+3)$ we can not deduce the continuity from ① - ③.

More generally, (Range of f defined on I)

If $f \in C(I)$ and $g \in C(f(I))$, then, the function $g \circ f$ defined by

$I \ni x \longrightarrow g(f(x))$

is continuous.

Reminder

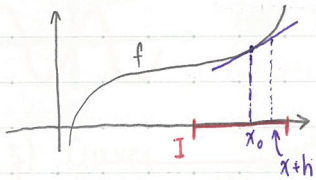
- graphs and curves
- limits at x_0 (if they are the same;)

$$\lim_{\substack{x \to x_0 \\ x \to x^+}} f(x), \quad \lim_{\substack{x \to x_0 \\ x \to x^-}} f(x), \quad \lim_{x \to x_0} f(x)$$

- continuity at one point

continuity on an interval $\rightsquigarrow C(I)$

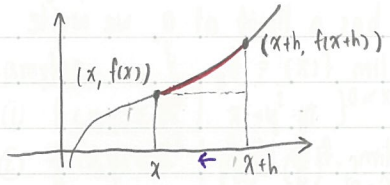
III.2) Slope of a function at a point



Let $f : I \longrightarrow \mathbb{R}$ and let $x, x+h \in I$

(I an open interval)

$\Rightarrow (x, f(x))$ and $(x+h, f(x+h))$ belong to the graph of f.



The slope of f at x is given by

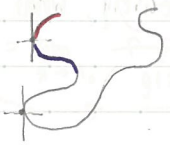
$$\frac{f(x+h)-f(x)}{(x+h)-(x)} = \frac{f(x+h)-f(x)}{h}$$ as h goes to 0 if this limit exists.

Def: Let $f : I \to \mathbb{R}$ an $x \in I$. The slope at x is given $\lim_{h \to 0} \frac{f(x+h)-f(x)}{h}$ if this limit exists.

Remarks.

- This slope has a geometric interpretation: it corresponds to the slope of the tangent of f at x

- We can define the slope at almost every point of a curve, by working locally.



(the slope is not defined when it is equal to "$\pm\infty$")

* critical point

III.3) The derivative of a function

Def: From $f:(a,b)\to\mathbb{R}$, and for $x\in(a,b)$ the limit $\lim_{h\to 0}\frac{f(x+h)-f(x)}{h}$ is called the "derivative of f at x", if it exists.

We write $f'(x)$ or $\frac{df(x)}{dx}$ for the derivative (which is a number)

If $f'(x)$ exists for all $x\in(a,b)$, we call "the derivative of f" the function.

$$f':(a,b)\to\mathbb{R}$$
$$x\mapsto f'(x)$$

If f' exists, we say the f is "differentiable".

Examples: consider

① $f:\mathbb{R}\to\mathbb{R}$, $f(x)=x^2$

One has $\frac{f(x+h)-f(x)}{h}$

$=\frac{(x+h)^2-x^2}{h}=\frac{2xh+h^2}{h}=2x+h\xrightarrow{h\to 0}2x$

Since $\frac{f(x+h)-f(x)}{h}$ has a limit for $h\to a$ and for any $x\in\mathbb{R}$, f is differentiable with derivative f' given by $f'(x)=2x$

② Consider $f:\mathbb{R}\to\mathbb{R}$, given $f(x)=|x|$.

One has $\frac{f(x+h)-f(x)}{h}=\frac{|x+h|-|x|}{h}$

$=\begin{cases}\frac{x+h-x}{h} & \text{if } x>0 \text{ and } |h| \text{ small enough}\\ \frac{-(x+h)-(-x)}{h} & \text{if } x<0 \text{ and } |h| \text{ small enough}\end{cases}$

$=\begin{cases}1 & \dots\\ -1 & \dots\end{cases}$

$\Rightarrow$ f is differentiable at x for any $x\neq 0$

⚠ For $x=0$, one has

$$\frac{f(0+h)-f(0)}{h}=\frac{|h|-0}{h}$$

$$=\begin{cases}1 & \text{if } h>0\\ -1 & \text{if } h<0\end{cases}$$

$\Rightarrow$ Since $\lim_{h\to 0^-}\frac{f(0+h)-f(h)}{h}=1$

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$\neq -1 = \lim_{h \to 0^-} \frac{f(0+h) - f(0)}{h}$

then f has no derivative at 0.

Remark

$\lim_{h \to 0} \dots$ means ε, δ !

More precisely, f is differentiable at x if $\exists\, f'(x) \in \mathbb{R}$ such that for any $\varepsilon > 0$, there exists $\delta > 0$ with

$$\left| \frac{f(x+h) - f(x)}{h} - f'(x) \right| \leq \varepsilon \qquad |h| \leq \delta$$

for any $h \in [-\delta, \delta]$

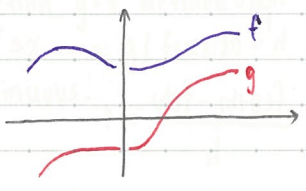
Def : If $f: (a,b) \to \mathbb{R}$ is differentiable on (a,b) and f' is continuous, we write $f \in C^1((a,b))$

↳ $\Leftrightarrow$ f, f' continuous

III.4) Back to limits.

Suppose $f, g: \mathbb{R}^* \to \mathbb{R}$ are continuous, and assume that

$$\lim_{x \to 0} f(x) = f(0)$$
$$\lim_{x \to 0} g(x) = g(0)$$



Then we know that

- $\lim_{x \to 0} (f+g)(x) = f(0) + g(0)$
- $\lim_{x \to 0} (fg)(x) = f(0)\, g(0)$
- $\lim_{x \to 0} \frac{f}{g}(x) = \frac{f(0)}{g(0)}$

What happen if $f(0) = 0 = g(0)$, what is $\frac{f(0)}{g(0)}$? Or more precisely, what $\lim_{x \to 0} \frac{f(x)}{g(x)} = ?$

Examples

① $f(x) = x^2$, $g(x) = x^2 + x$

then $\frac{f(x)}{g(x)} = \frac{x^2}{x^2+x} = \frac{x}{x+1} \xrightarrow{x \to 0} 0.$

② $f(x) = x^2$, $g(x) = x^3$

$\frac{f(x)}{g(x)} = \frac{x^2}{x^3} = \frac{1}{x}$ has no limit as $x \to 0$.

③ $f(x) = x^2$, $g(x) = x^2 + x^3$,

then $\frac{f(x)}{g(x)} = \frac{x^2}{x^2+x^3} = \frac{1}{1+x} \xrightarrow{x \to 0} 1$

L'Hospital's Rule

Let $f, g: (a,b) \to \mathbb{R}$ be differentiable on (a,b).

Let $x_0 \in (a,b)$, and assume $f(x_0) = 0 = g(x_0)$

Assume also that $g'(x) \neq 0$ for any $x \in (a,b) \setminus \{x_0\}$.

Then $\lim_{x\to x_0} \frac{f(x)}{g(x)} = \lim_{x\to x_0} \frac{f'(x)}{g'(x)}$ if this limit exists.

Proof. (we assume that $g'(x_0) \neq 0$ — skip this assumption later on)

One has $\lim_{x\to x_0} \frac{f(x)-f(x_0)}{g(x)-g(x_0)}$ we can subtract because both are 0.

$$= \lim_{x\to x_0} \frac{\frac{f(x)-f(x_0)}{x-x_0}}{\frac{g(x)-g(x_0)}{x-x_0}} \qquad \frac{f(x_0+h)-f(x_0)}{h} \text{ if } h = x-x_0$$

$$= \frac{\lim_{x\to x_0} \frac{f(x)-f(x_0)}{x-x_0}}{\lim_{x\to x_0} \frac{g(x)-g(x_0)}{x-x_0}} = \frac{f'(x_0)}{g'(x_0)}$$

$$= \lim_{x\to x_0} \frac{f(x)}{g'(x)} \qquad \#$$

III.5) Properties of differentiation

Let $f, g : I \to \mathbb{R}$ be differentiable on I, and let $\lambda \in \mathbb{R}$. Then

1) $(\lambda f)'(x) = \lambda f'(x)$
2) $(f+g)'(x) = f'(x) + g'(x)$
3) $(fg)'(x) = f'(x)g(x) + f(x)g'(x)$
4) $\left(\frac{f}{g}\right)' = \frac{f'(x)g(x) - f(x)g'(x)}{g^2(x)}$ if $g(x) \neq 0$.

$\forall x \in I$

Proof.

1) $$\frac{(\lambda f)(x+h) - (\lambda f)(x)}{h}$$
$$= \frac{\lambda f(x+h) - \lambda f(x)}{h}$$
$$= \lambda \frac{f(x+h)-f(x)}{h} \xrightarrow{h\to 0} \lambda f'(x)$$

2) $$\frac{(f+g)(x+h) - (f+g)(x)}{h}$$
$$= \frac{f(x+h)+g(x+h)-f(x)-g(x)}{h}$$
$$= \frac{f(x+h)-f(x)}{h} + \frac{g(x+h)-g(x)}{h}$$
$$\xrightarrow{h\to 0} f'(x) + g'(x)$$

3) $$\frac{(fg)(x+h) - (fg)(x)}{h}$$
$$= \frac{f(x+h)g(x+h) - f(x)g(x)}{h}$$
$$= \frac{f(x+h)\left(g(x+h)-g(x)\right) + f(x+h)g(x) - f(x)g(x)}{h}$$
$$= f(x+h)\frac{g(x+h)-g(x)}{h} + \frac{f(x+h)-f(x)}{h}g(x)$$
$$\xrightarrow{h\to 0} f(x)g'(x) + f'(x)g(x)$$

A differentiable func. is continuous.

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4) $$\frac{\frac{f(x+h)}{g(x+h)} - \frac{f(x)}{g(x)}}{h}$$

$$= \frac{1}{h} \frac{f(x+h)g(x) - g(x+h)f(x)}{g(x+h)g(x)}$$

$$= \frac{\frac{f(x+h)-f(x)}{h} g(x) - \frac{g(x+h)-g(x)}{h} f(x)}{g(x+h)g(x)}$$

$$\xrightarrow{h \to 0} \frac{f'(x)g(x) - g'(x)f(x)}{g(x)^2}$$

#

Something we are about to prove:

small
- Lemma
- Proposition
- Theorem

(Proposition, Theorem: get from the smaller ones)

big
- Corollary : almost obvious

Reminder:

- Derivatives
- L'Hospital's rule
- 4 properties for differentiation

Applications

1.) Consider $f: \mathbb{R} \to \mathbb{R}$ differentiable and $f(x) > 0$ $\forall x \in \mathbb{R}$

$$\left(\frac{1}{f}\right)' = -\frac{f'}{f^2}$$

2.) Consider $x \mapsto x^{-n} := \frac{1}{x^n}$

$$(x^{-n})' = \left(\frac{1}{x^n}\right)' = -\frac{n x^{n-1}}{x^{2n}} = -n \frac{1}{x^{n+1}}$$

$$= -n x^{-n-1}$$

Chain Rule:

Let $f, g : \mathbb{R} \to \mathbb{R}$, differentiable and consider $(f \circ g)(x) = f(g(x))$

$$(f \circ g)'(x) = f(g(x))' = f'(g(x))\, g'(x)$$

Lemma If $f, g : \mathbb{R} \to \mathbb{R}$ are differentiable, then $f(g(x))' = f'(g(x))\, g'(x)$

Proof: We consider

$$\frac{f(g(x+h)) - f(g(x))}{h}$$

set $u := g(x)$, $k := g(x+h) - g(x)$

$$= \frac{f(u+k) - f(u)}{k} \frac{k}{h}$$

$k = 0$? This can be 0, but the proof can be adapted.

$$= \frac{f(u+k) - f(u)}{k} \cdot \frac{g(x+h) - g(x)}{h}$$

$k \to 0 \Rightarrow f'(u)$ $\quad$ $h \to 0 \Rightarrow g'(x)$

$\Rightarrow \lim_{h\to 0} \frac{f(g(x+h)) - f(g(x))}{h}$

$= f'(u)\, g'(x)$

$= f'(g(x))\, g'(x)$ #

Observe that $k \to 0$ when $h \to 0$ since g is continuous.

Remark One often see

$$\frac{d f(g(x))}{dx} = \frac{df}{dg}\frac{dg}{dx}$$

▽ Prof. doesn't like this form. Use the other one please. ☺

Example: Consider

$$x \mapsto g(x)^3$$

f is given by $x \mapsto x^3$

Then $(g(x)^3)' = 3(g(x))^2 g'(x)$

III.6) Higher derivatives

Let $f : I \to \mathbb{R}$ be differentiable with $f' : I \to \mathbb{R}$

If f' is differentiable, then we can compute $(f')' \equiv f'' \equiv f^{(2)}$

Notations

$f^{(j)}$ is the j^{th} derivative of f.

If $f^{(j)}$ is continuous, we write $f \in C^j(I)$

Example: let $P_n : \mathbb{R} \to \mathbb{R}$ (polynomial degree n)

$$P_n(x) = x^n$$

Then $P_n' = n P_{n-1}$ equality of functions

$P_n^{(2)} = n(n-1) P_{n-2}$

$P_n^{(3)} = n(n-1)(n-2) P_{n-3}$

$\vdots$

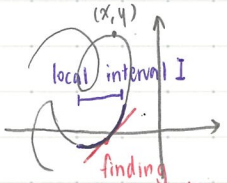
$P_n^{(n)} = n(n-1)(n-2)\dots 1 =: n!$

$P_n^{(n+1)} = 0$ function equal to 0 for any x. It may look like a zero, but it's a function. function = function

$$\left[(x^n)' = n x^{n-1} = n P_{n-1}(x) \right]$$

equality of numbers

III.7) Implicit differentiation



Often, curves can be described by an equation, it means a relation between x and y.

Example

$$x^7 y^6 + x^2 y^3 + x^{25} y + x y^{23} = 0$$

we can't solve this, but it described the curve. We know some points like $(0,0)$.

In general, we write $F(x,y) = 0$

with $F : \mathbb{R} \times \mathbb{R} \to \mathbb{R}$

x y $F(x,y)$

a vertical "ε"

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Example Consider the curve given by

$$F(x,y) = 3x^3y - y^4 + 5x^2 + 5 = 0$$

Suppose that we can solve this equation, and write $y = f(x)$ (at least locally). Then we can consider $F(x, f(x)) \equiv F(x, y(x))$ and take its derivative.

One has $\frac{d}{dx} F(x, f(x)) \equiv \frac{d}{dx} F(x, y(x))$

$$= 9x^2 y(x) + 3x^3 y'(x) - 4y^3(x)\, y'(x) + 10x = 0$$

We need this to find the slope.

because $F(x, y(x)) = 0$ $\forall x \in I$

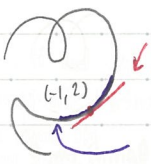
We get $(3x^3 - 4y(x)^3)\, y'(x) = -9x^2 y(x) - 10x$

$$\Rightarrow \quad \boxed{y'(x) = \frac{9x^2 y(x) + 10x}{4y(x)^3 - 3x^3}}$$

Observe that $(1,2)$ satisfies $F(1,2) = 0$, which means that $(1,2)$ belongs to the curve.

$\Rightarrow$ $y(1) = 2$ and we get

$$y'(1) = \frac{18 + 10}{32 - 3} = \frac{28}{29}$$



slope of this tangent is $y'(1) = \frac{28}{29}$

So we can get the slope without knowing the (local) function YAY!

IV.) Some Basic Functions

IV.1) Sine and cosine functions

$\sin, \cos : \mathbb{R} \to [-1,1]$, 2π periodic and satisfy

- $\cos^2(x) + \sin^2(x) = 1$
- $\sin(x+y) = \sin(x)\cos(y) + \cos(x)\sin(y)$
- $\cos(x+y) = \dots$
- $\sin(x) - \sin(y) = 2\sin\left(\frac{x-y}{2}\right)\cos\left(\frac{x+y}{2}\right)$
- $\cos(x) - \cos(y) = \dots$

Lemma: $\lim_{h \to 0} \frac{\sin(h)}{h} = 1$

Corollary: $\sin(x)' \equiv \sin'(x) = \cos(x)$

Proof: $\sin'(x) = \lim_{h \to 0} \frac{\sin(x+h) - \sin(x)}{h}$

$$= \lim_{h \to 0} \frac{2\sin\left(\frac{h}{2}\right)\cos\left(x + \frac{h}{2}\right)}{h}$$

$$= \lim_{h \to 0} \frac{\sin\left(\frac{h}{2}\right)}{\frac{h}{2}} \cos\left(x + \frac{h}{2}\right)$$

from lemma $\to 1$ ($h \to 0$); $\cos(x)$ ($h \to 0$)

$$= \lim_{h \to 0} \frac{\sin\left(\frac{h}{2}\right)}{\frac{h}{2}} \cdot \lim_{h \to 0} \cos\left(x + \frac{h}{2}\right)$$

$$= 1 \cdot \cos(x)$$

$$= \cos(x) \qquad \#$$

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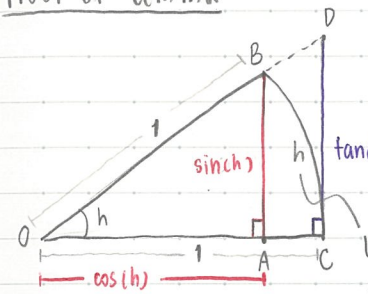
Corollary (Homework)

1.) $\cos'(x) = -\sin(x)$

2.) $\lim_{x \to 0} \frac{\cos(x)-1}{x} = 0$

3.) $\lim_{x \to 0} \frac{\cos(x)-1}{x^2} = -\frac{1}{2}$

Proof of Lemma



Consider $h > 0$ (for simplicity)

$\tan(h) = \frac{\sin(h)}{\cos(h)}$

length equals to h (in radian)

One has $\Delta OAB \leq$ $\leq \Delta OCD$

$\Leftrightarrow \frac{1}{2}\cos(h)\sin(h) \leq \frac{h}{2\pi} \cdot \pi \cdot 1^2 \leq \frac{1}{2}\tan(h) = \frac{1}{2}\frac{\sin(h)}{\cos(h)}$

fraction of the disk

area of the disk

$\Leftrightarrow \cos(h)\sin(h) \leq h \leq \frac{\sin(h)}{\cos(h)}$

$\Leftrightarrow \cos(h) \leq \frac{h}{\sin(h)} \leq \frac{1}{\cos(h)}$

$\Leftrightarrow \frac{1}{\cos(h)} \geq \frac{\sin(h)}{h} \geq \cos(h)$

Since $\lim_{h \to 0} \cos(h) = 1$

We infer that $\lim_{h \to 0} \frac{\sin(h)}{h} = 1$ #.

(squeeze theorem)

IV.2) Exponential functions

In Homework 4 we define

$$f : \mathbb{R} \to \mathbb{R}, \quad f(x) = \sum_{j=0}^{\infty} \frac{1}{j!} x^j$$

and we obtain that

$$f'(x) = f(x)$$

We call f the exponential, and write

$$f(x) = e^x$$

name of the function defined by

Lemma Let $g : \mathbb{R} \to \mathbb{R}$ be a differentiable function satisfying $g'(x) = g(x)$.

Then $g(x) = ce^x$ for some $c \in \mathbb{R}$

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Reminder

- chain rule = differentiable for composition
- higher order derivatives, $C^n(\mathbb{R})$
- implicit differentiation
- $\lim_{h\to 0} \frac{\sin(h)}{h} = 1 \rightsquigarrow \sin'(x) = \cos(x)$
- $e^x := \sum_{n=0}^{\infty} \frac{x^n}{n!}$, $e^{x\prime} = e^x$

Lemma If $f: \mathbb{R} \to \mathbb{R}$ differentiable satisfies $f' = f$, then $f(x) = ce^x$ for some $c \in \mathbb{R}$

Proof Let us show that $\frac{f(x)}{e^x}$ is a constant independent of x

Indeed, one has

$$\left(\frac{f(x)}{e^x}\right)' = \frac{f'(x)e^x - (e^x)f(x)}{(e^x)^2}$$

($(e^x)' = e^x$)

$$= \frac{f(x)e^x - f(x)e^x}{(e^x)^2} = 0$$

($f(x)$ assumption)

Since $\left(\frac{f(x)}{e^x}\right)' = 0$, one infers that

$\frac{f(x)}{e^x} = c$ for some $c \in \mathbb{R}$

and then $f(x) = ce^x \quad \forall x \in \mathbb{R}$ #

Remark We will show that $e^x \neq 0$ for any $x \in \mathbb{R}$

Note

$$\sum_{n=0}^{N} a^n = \frac{1-a^{N+1}}{1-a} \qquad (a \neq 1)$$

$$(1-a)\sum_{n=0}^{N} a^n = (1-a)(a^0 + a^1 + a^2 + \dots + a^N)$$

$$= a^0 + a^1 + a^2 + \dots + a^N - a^1 - a^2 - a^3 - \dots - a^N - a^{N+1}$$

$$= a^0 - a^{N+1}$$

$$= 1 - a^{N+1}$$

$$\therefore \sum_{n=0}^{N} a^n = \frac{1-a^{N+1}}{1-a}$$

If $|a| < 1$; $\sum_{n=0}^{\infty} a^n = \frac{1}{1-a}$

I.) Mean Value Theorem

I.1) Local minimum and maximum

Def: Let $f: I \to \mathbb{R}$. A point $x_0 \in I$ is a local maximum for f if $\exists \delta > 0$ such that $f(x_0) \geq f(x_0 + h)$ for and $|h| < \delta$
$\iff f(x_0) \geq f(y)$ for any $y \in [x_0 - \delta, x_0 + \delta]$

x_0 is a local minimum for f. if $\exists \delta > 0$ such that $f(x_0) \leq f(x_0 + h)$ for any $|h| < \delta$,

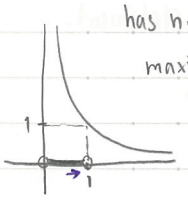
x_0 is a local maximum of f if
$f(x_0) \geq f(y) \quad \forall y \in I$ (with $x_0 + h \in I$)

x_0 is a global minimum of f if
$f(x_0) \leq f(y) \quad \forall y \in I$

Question: Do they alway exist ? and are they unique ?

Ans: NO. For uniqueness:
local $\neq$ global maximum

$\frac{1}{x}$ on $(0,1)$ ← open interval
has no local or global maximum (and also minimum)



But on $(0,1]$, the function $x \mapsto \frac{1}{x}$ has a a local (and global) minimum at $x = 1$.

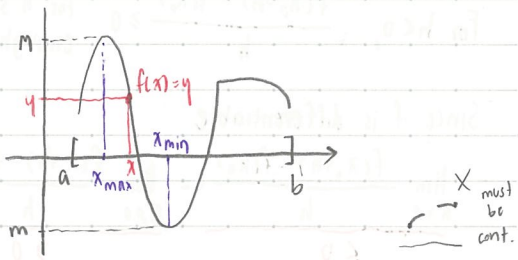
Thm Extreme Value thm.

If $f: [a,b] \to \mathbb{R}$ is continuous, then there exists $x_{min} \in [a,b]$ and $x_{max} \in [a,b]$ with x_{min} a global minimum, and x_{max} a global maximum.

⚠ They might not be unique

Thm Intermidiate value thm

Let $f: [a,b] \to \mathbb{R}$ be continuous, and set $m := f(x_{min})$ and $M := f(x_{max})$. Then for any $y \in [m, M]$ there exist $x \in [a,b]$ with $f(x) = y$. x might not be unique



Def Let $f: I \to \mathbb{R}$ be differentiable
A point $x_0 \in I$ is a critical point for f if $f'(x_0) = 0$

examples.

1) $f(x) = \sin(x)$, $f'(x) = \cos(x)$, $f: \mathbb{R} \to \mathbb{R}$
and $\cos(x) = 0 \iff x = \frac{\pi}{2} + k\pi$, $k \in \mathbb{Z}$
Critical point $= \{\frac{\pi}{2} + k\pi \mid k \in \mathbb{Z}\}$

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2) $f: \mathbb{R} \to \mathbb{R}$

$f(x) = x^3$

$f'(x) = 3x^2$

$3x^2 = 0 \iff x = 0$

critical point = $\{0\}$

(a,b) open

Thm Let $f: I \to \mathbb{R}$ be differentiable, and let $x_0 \in I$ with x_0 a local maximum (minimum), Then $f'(x_0) = 0$

(which means x_0 is a critical point for f)

Proof: Consider $h \in \mathbb{R}$, small with $x_0 + h \in I$

For $h > 0$, $\dfrac{f(x_0+h) - f(x_0)}{h} \leq 0$ for h small enough

For $h < 0$, $\dfrac{f(x_0+h) - f(x_0)}{h} \geq 0$ for h small enough

Since f is differentiable,

$$\lim_{h \searrow 0} \frac{f(x_0+h) - f(x_0)}{h} = \lim_{h \nearrow 0} \frac{f(x_0+h) - f(x_0)}{h}$$

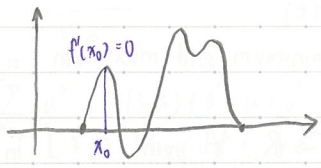
≤ 0 $\qquad$ ≥ 0

which implies that $\lim_{h \to 0} \dfrac{f(x_0+h) - f(x_0)}{h} = 0$

$\| \; f'(x)$

Do the same for local minimum. #

V.2) The mean value theorem



Thm (Rolle's Thm.)

Let $f: [a,b] \to \mathbb{R}$, continuous, and differentiable on (a,b).

Suppose $f(a) = f(b) = 0$. Then there exists $x_0 \in (a,b)$ with $f'(x_0) = 0$.

$f: [0,1)$ continuous at 0, but not differentiable at 0.

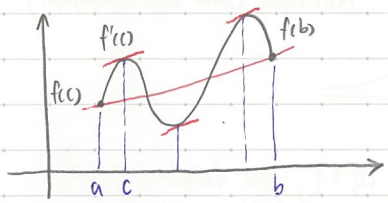
example: $|x|$, $\sqrt{x}$

proof

- If $f(x) = 0 \quad \forall x \in [a,b]$, then $f' = 0$, the statement is satisfied for any $x \in (a,b)$
- suppose $\exists x \in (a,b)$ with $f(x) > 0$ (or < 0)

Then, by the extreme value thm, $\exists x_{max} \in (a,b)$ which is a global maximum.

By the previous thm, $f'(x_{max}) = 0$, which proves the statement. #



slope $= \dfrac{f(b) - f(a)}{b - a} = f'(c)$

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Thm (mean value thm.) cont. diff. Let $f:[a,b]$ be continuous, and differentiable on (a,b). Then $\exists c \in (a,b)$ with

$$f'(c) = \frac{f(b)-f(a)}{b-a}$$

Proof Equation of the line:

$$y = \frac{f(b)-f(a)}{b-a}(x-a)+f(a)$$

consider $h:[a,b] \to \mathbb{R}$,

$$h(x) = f(x) - \left(\frac{f(b)-f(a)}{b-a}(x-a)+f(a)\right)$$

continue next class.

(continue)

One $h(a)=0$, $h(b)=0$

h is cont. on $[a,b]$, diff on (a,b)

By Rolle's thm, $\exists c \in (a,b)$ with $h'(c)=0$

But $h'(x) = f'(x) - \frac{f(b)-f(a)}{b-a}$

$$\Rightarrow h'(c) = f'(c) - \frac{f(b)-f(a)}{b-a} = 0$$

$$\Rightarrow f'(c) = \frac{f(b)-f(a)}{b-a}$$

Reminder

- local / global max / min
- extreme value thm.
- intermediate value thm.
- critical point
- Rolle's thm.
- mean value thm.

Corollary Let $f: I \to \mathbb{R}$ be diff. and such that $f'(x)=0 \quad \forall x \in I$.
Then f = constant. const.

Proof: Choose $a,b \in I$ interval with $a<b$
Then $f:[a,b] \to \mathbb{R}$ is continuous and diff. on (a,b)
By the mean value thm, $\exists c \in (a,b)$ with

$$f'(c) = \frac{f(b)-f(a)}{b-a}$$

Since $f'(c)=0$, then $f(b)=f(a)$
$\Rightarrow$ f takes the same value.
On any pair of points in I
$\Rightarrow$ f = const. #

New proof of L'Hospital's Rule:

Step 1: Cauchy mean value thm.

Lemma: Let $f,g:[a,b] \to \mathbb{R}$ be cont. and diff. on (a,b). Then $\exists c \in (a,b)$ with

$$f'(c)(g(b)-g(a)) = g'(c)(f(b)-f(a)) \quad (*)$$

Proof 1) if $g(b) = g(a)$, then we can apply Rolle's thm. to $x \mapsto g(x) - g(x)$, and get $g'(c) = 0$ for some $c \in (a,b)$

2) if $g(b) \neq g(a)$, set $h: [a,b] \to \mathbb{R}$

$$h(x) = f(x) - \frac{f(b)-f(a)}{g(b)-g(a)} g(x) - \frac{f(a)f(b) - f(b)f(a)}{g(b)-g(a)}$$

Then h is cont. on $[a,b]$ and diff. on (a,b). One has $h(a) = 0 = h(b)$. By Rolle's thm, $\exists c \in (a,b)$ with $h'(c) = 0$

$$\Leftrightarrow f'(c) - \frac{f(b)-f(a)}{g(b)-g(a)} g'(c) = 0 \Rightarrow (*)$$ #

Step 2: proof of L'Hospital's thm.

$$\frac{f(x)}{g(x)} = \frac{f(x) - f(x_0)}{g(x) - g(x_0)} = \frac{f'(c)}{g'(c)}$$

↑ by step 1, when $c \in (x_0, x)$

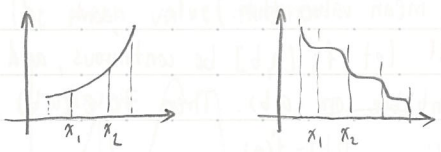
$$\Rightarrow \lim_{x \to x_0} \frac{f(x)}{g(x)} = \lim_{x \to x_0} \frac{f'(c)}{g'(c)} = \lim_{c \to x_0} \frac{f'(c)}{g'(c)}$$

if the limit exists. #

Def A function $f: I \to \mathbb{R}$ is increasing if $f(x_1) \leq f(x_2)$ whenever $x_1 \leq x_2$.

And f is decreasing if $f(x_1) \geq f(x_2)$ whenever $x_1 \leq x_2$.

We say that it is strictly increasing / decreasing if $f(x_1) < f(x_2)$ or $f(x_1) > f(x_2)$ for $x_1 < x_2$.



Lemma Let $f: [a,b] \to \mathbb{R}$ be cont. and diff. on (a,b).

1.) If $f'(x) \geq 0 \quad \forall x \in (a,b)$, then f is increasing on $[a,b]$.

2) If $f'(x) \leq 0 \quad \forall x \in (a,b)$, the f is decreasing on $[a,b]$

Proof of 1)

Choose $x_1, x_2 \in (a,b)$ with $x_1 < x_2$. Then by the mean value thm. $\exists c \in (x_1, x_2)$ with

$$0 \geqslant 0 \leqslant f'(c) = \frac{f(x_2) - f(x_1)}{x_2 - x_1}$$

$$\Rightarrow f(x_2) - f(x_1) \overset{\leqslant}{\geqslant} 0$$

$$\Leftrightarrow f(x_2) \overset{\leqslant}{\geqslant} f(x_1)$$

Similar proof for 2.)

VI.) Sketching Curves.

look at the website.

VII.) Inverse Function

VII. 1) Definition of inverse function

Let $f: I \to \mathbb{R}$ and set $y = f(x)$

Question: Can we solve $x = g(y)$ for some g?

More precisely, can we find $g: \ldots \to I$ such that
$g(f(x)) = x \quad \forall x \in I$
$f(g(y)) = y \quad \forall y \in f(I)$ or $\mathrm{Ran}(f)$

Examples:

① $f: \mathbb{R}_+ \to \mathbb{R}_+, \quad f(x) = x^2$
If we set $g: \mathbb{R}_+ \to \mathbb{R}_+$
$x \mapsto \sqrt{x} = g(x)$
then $g(f(x)) = x \quad \forall x \in \mathbb{R}_+$
$f(g(y)) = y \quad \forall y \in \mathbb{R}_+$

② $f: \mathbb{R} \to \mathbb{R}$
$x \mapsto x^2$
ex. $x = 3$ and -3 have the same y

We cannot find an inverse because f is not injective.

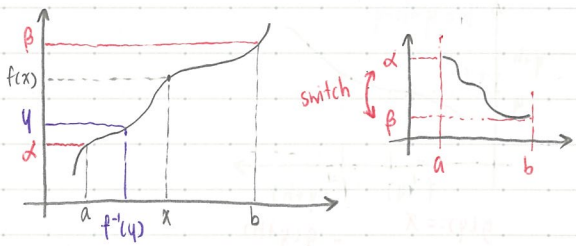
Remark

Since f is always surjective on its image (on $f(I)$ or $\mathrm{Ran}(f)$) then $f: I \to f(I)$ is bijective if and only if f is injective.

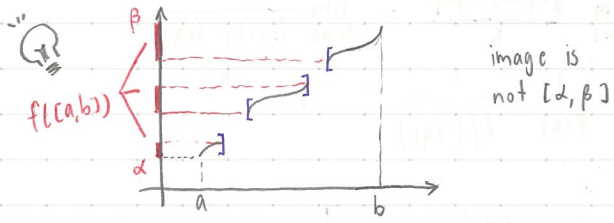
In such a case, we can define an inverse.

Thm: Let $f: [a,b] \to \mathbb{R}$ be strictly increasing and cont. Set $\alpha := f(a)$ and $\beta := f(b)$. Then there exists
notation (f^{-1}): $[\alpha, \beta] \to [a,b]$ with
$f^{-1}(f(x)) = x \quad \forall x \in [a,b]$ and
$f(f^{-1}(y)) = y \quad \forall y \in [\alpha, \beta]$

Proof as an exercise.



Similar statement if f is strictly decreasing.



VII. 2) Derivative of the inverse function

Thm. Let $f: [a,b] \to \mathbb{R}$ be differentiable on (a,b) with $f'(x) > 0 \quad \forall x \in (a,b)$.
Set $\alpha := f(a)$, $\beta := f(b)$
Then $f^{-1}: [\alpha, \beta] \to [a,b]$ is diff. and

$$(f^{-1})'(y) = \frac{1}{f'(f^{-1}(y))}$$

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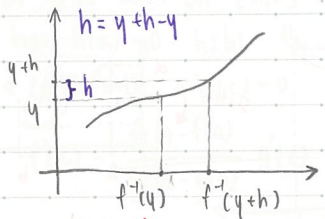
Proof Set $g(y) := f^{-1}(y)$. One has

$$\frac{g(y+h)-g(y)}{h} = \frac{x+k-x}{f(x+k)-f(x)}$$

$$= \frac{1}{\frac{f(x+k)-f(x)}{k}}$$

when $h \to 0$, then $k \to 0$.



$g(y) := x$ $\quad = g(y+h)$ $\quad = x+k$

$$\lim_{h\to 0} \frac{g(y+h)-g(y)}{h} = \lim_{h\to 0} \frac{1}{\frac{f(x+k)-f(x)}{k}}$$

$$= \frac{1}{f'(x)} = \frac{1}{f'(f^{-1}(y))}$$

Thus, $(f^{-1})'(y) = \frac{1}{f'(f^{-1}(y))}$ #

Reminder

- mean value thm.
- proof of L'Hospital's thm
- increasing / decreasing function
- If f differentiable, $f' \geq 0 \Rightarrow f$ increasing.
  $f' \leq 0 \Rightarrow f$ decreasing.
- Inverse function
- $(f^{-1})'(y) = \frac{1}{f'(f^{-1}(y))}$

Examples

1.) $\sin: [-\frac{\pi}{2}, \frac{\pi}{2}] \ni x \mapsto \sin(x) \in [-1, x]$

$\sin'(x) = \cos(x) > 0$ on $(-\frac{\pi}{2}, \frac{\pi}{2})$

since sin is invertible, we can consider its inverse, $\sin^{-1} =: \arcsin$ and

$$\arcsin'(y) = \frac{1}{\cos(\arcsin(y))} = \frac{1}{\sqrt{1-\sin(\arcsin(y))^2}}$$

$$= \frac{1}{\sqrt{1-y^2}}$$ for any $y \in (-1,1)$

2.) $\cos: [0, \pi] \ni x \mapsto \cos(x) \in [-1, 1]$

$\cos'(x) = -\sin(x) < 0$ for $x \in (0, \pi)$

since cos is invertible, we set

$\cos^{-1} = \arccos : (-1,1) \mapsto (0, \pi)$ with

$$\arccos(y') = \frac{1}{-\sin(\arccos(y))} = -\frac{1}{\sqrt{1-y^2}}$$

3.) $\tan: (-\frac{\pi}{2}, \frac{\pi}{2}) \ni x \mapsto \tan(x) \in \mathbb{R}$

$\tan^{-1}(x) = 1 + \tan^2(x) \geq 1 > 0$

tan is invertible, and $\tan^{-1} = \arctan : \mathbb{R} \to (-\frac{\pi}{2}, \frac{\pi}{2})$

$$\arctan'(y) = \frac{1}{1+\tan(\arctan(y))^2} = \frac{1}{1+y^2}$$

VII.3) Exponential and Logarithmic functions

Reminder $e^x := \sum_{n=0}^{\infty} \frac{1}{n!} x^n$, and $e^{x'} = e^x$, $e^0 = 1$

Remark : $e^{-x} := \frac{1}{e^x} \overset{?}{\neq} e^{-x}$

Lemma : Let us set $f(x) := \sum_{n=0}^{\infty} \frac{1}{n!} x^n$

1) $f(x) f(-x) = 1 \quad \forall x \in \mathbb{R}$
2) $f(x) > 0 \quad \forall x \in \mathbb{R}$
3) f is increasing
4) $f(x+y) = f(x) f(y) \quad \forall x, y \in \mathbb{R}$

Proof

1.) $(f(x)f(-x))' = f'(x)f(-x) + f(x)(f(-x))'$
$= f(x)f(-x) + f(x) \underbrace{f'(-x)}_{= f(-x)} (-1) = 0$
$\Rightarrow x \mapsto f(x)f(-x)$ is a constant function since $f(0) = 1 = f(-0)$
$\Rightarrow f(x)f(-x) = 1 \quad \forall x \in \mathbb{R}$

2.) It follows from 1) that $f(x) \neq 0$ for any x (otherwise $f(x)f(-x) \neq 1$)
By the intermediate value thm, if there exists $x \in \mathbb{R}$ such that $f(x) < 0$, the $[f(x), 1]$ is in the range of f
$\Rightarrow 0 \in \mathrm{Ran}(f)$, which is not possible.

3.) $f'(x) = f(x) > 0 \Rightarrow f$ is increasing

4.) For fixed x, consider the function
$\phi : \mathbb{R} \to \mathbb{R}$, $\phi(y) := \frac{f(x+y)}{f(y)}$

($f(x+y)$ is a composition of two functions. $f(x+y) = f(g(x))$ with $g(x) = x+y$.)

One has $\phi'(y) = \frac{f'(x+y)f(y) - f(x+y)f'(y)}{f(y)^2} = 0$

because $f' = f \Rightarrow y \mapsto \phi(y)$ is a constant function.

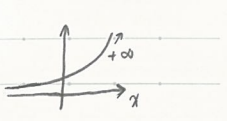
By considering $\phi(0) = \frac{f(x+0)}{f(0)} = f(x)$
we get $\phi(y) = f(x)$
$\Leftrightarrow f(x+y) = f(x)f(y)$ #.

Remark : If we use the notation $f(x) = e^x$ we have obtained :

① $e^x e^{-x} = 1$ $\quad (\Rightarrow e^{-x} = \frac{1}{e^x})$
② $e^x > 0$
③ $x \mapsto e^x$ is increasing
④ $e^{x+y} = e^x e^y$

We call this function the "exponential function".

Since $(e^x)' = e^x > 0$
$\Rightarrow e^x$ is strictly increasing, and then it is invertible. Its inverse in denoted by $\ln : \mathbb{R}^+ \to \mathbb{R}$ with $\ln(e^x) = x \quad \forall x \in \mathbb{R}$ and $e^{\ln(y)} = y \quad \forall y \in \mathbb{R}^+$



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Lemma: (HW7.)

1.) $\ln(y)' = \frac{1}{y}$

2.) $\ln(xy) = \ln(x) + \ln(y)$ } $\forall x, y \in \mathbb{R}^+$

3.) $\ln(y^x) = x \ln(y)$ $\forall y \in \mathbb{R}^+, x \in \mathbb{Q}$

Remark

Since for any $x \in \mathbb{R}$, there exists a sequence $(x_n)_n \in \mathbb{N}$ with $x_n \in \mathbb{Q}$ and $x_n \xrightarrow{n\to\infty} x$, we would like to have:

$$\ln y^x \overset{!}{=} \ln y^{\lim_{n\to\infty} x_n} \overset{!}{=} \lim_{n\to\infty} \ln y^{x_n}$$

$$= \lim_{n\to\infty} x_n \ln y = x \ln(y)$$

↑ because $x_n \in \mathbb{Q}$

Def: For any $y \in \mathbb{R}^+$ and $x \in \mathbb{R}$ one set $y^x = e^{\ln(y^x)} := e^{x \ln y}$

$\Rightarrow \ln(y^x) = \ln(e^{x \ln y})$

$= x \ln y$.!

↑ midterm is everything until this point

Good luck !!

☺

VIII.) Integration

1.) We would like to find an inverse operation for the differentiation

2.) Compute some area

VIII.1) Indefinite integral

Prof. doesn't like this word.

= antiderivative

Def: Let $f: I \to \mathbb{R}$ be a function. An indefinite integral for f is a function $F: I \to \mathbb{R}$, differentiable with $F' = f$

Examples

① $f: \mathbb{R} \to \mathbb{R}$, $f(x) = x^n$ $n \in \mathbb{N}$

then $F(x) = \frac{1}{n+1} x^{n+1} + k$ with $k \in \mathbb{R}$

② $f: \mathbb{R} \to \mathbb{R}$, $f(x) = \cos(x)$, $F(x) = \sin(x)$

Remarks:

1.) If F is an indefinite integral for f, then $F+k$, $k \in \mathbb{R}$, is also an indefinite integral

2.) We set $\int f = \int f(x)dx$ for an independent integral.

⚠ If we look for an indefinite integral of f on a domain made of more than 1 piece, one has to add several constants.

Example: consider $f: \mathbb{R}^* \to \mathbb{R}$ (exclude zero)

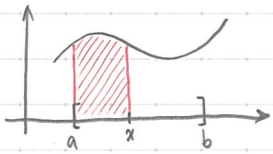
$$x \mapsto \frac{1}{x}$$

What is the most general indef. int. (indefinite integral) for f?

$$F(x) := \begin{cases} \ln(x) + k_1 & \text{for } x > 0 \\ \ln(-x) + k_2 & \text{for } x < 0 \end{cases}$$

VIII.2) Some areas

Consider $f: [a,b] \to \mathbb{R}^+$, continuous. For any $x \in [a,b]$, set $F(x) :=$ area under the graph of f between a and x.



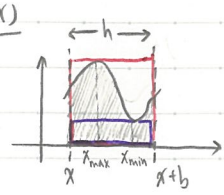
Observe that $F(a) = 0$

Thm.: The function $F: [a,b] \to \mathbb{R}$ is differentiable, and $F'(x) = f(x)$, $\forall x \in (a,b)$

Remark: The statement cannot be very precise because we have not defined what is the area under the curve.

Proof. Consider $x \in (a,b)$ and $h \in \mathbb{R}$ with h small enough such that $x+h \in (a,b)$

Consider $\frac{F(x+h) - F(x)}{h}$ and consider



$x_{max} \in [x, x+h]$

$x_{min} \in [x, x+h]$

$$h f(x_{min}) \leq F(x+h) - F(x) \leq h f(x_{max})$$

$$\Leftrightarrow f(x_{min}) \leq \frac{F(x+h)-F(x)}{h} \leq f(x_{max})$$

When $h \to 0$, x_{min} and x_{max} converge to x and then

$$f(x) \leq \lim_{h \to 0} \frac{F(x+h)-F(x)}{h} \leq f(x) \quad \#$$

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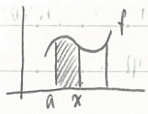
Reminder

- Indefinite integral (not unique)
- $f: [a,b] \to \mathbb{R}^+$, continuous.

and $F(x) =$ 

Then $F: [a,b] \to \mathbb{R}$, differentiable, and $F' = f$.

Remark

The function F is an indefinite integral for f on (a,b).

The area under the curve between a and b is given by $F(b) = F(b) - F(a)$

Corollary Let $f: [a,b] \to \mathbb{R}^+$ be continuous, and let $G: [a,b] \to \mathbb{R}$ be an indefinite integral for f for (a,b). Then the area under the curve between a and b is given by $G(b) - G(a)$.

Proof Let f be the function defined by the previous thm,

Then $G = F + \text{const.}$ (see previous lemma)

$\Rightarrow G(b) - G(a) = F(b) + \text{const.} - (F(a) + \text{const.})$

$= F(b)$

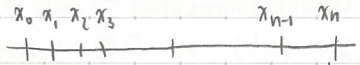
VIII.3) Riemann's Sum

Question: How can one compute the surface below the curve more precisely.

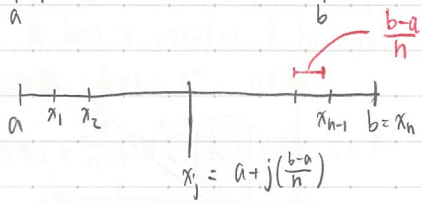
Def For any interval $[a,b]$, we call a n-partition $\mathcal{P}$ of $[a,b]$ a collection

$\{x_0, x_1, \dots, x_n\} = \{x_j\}_{j=0}^n$

with $x_0 = a$, $x_n = b$ and $x_j < x_{j+1}$



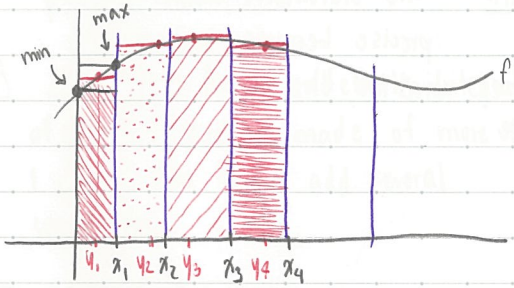
Example



Def For any $f: [a,b] \to \mathbb{R}$, bounded for any n-partition $\mathcal{P}$, and for any family $\{y_j\}_{j=1}^n$ with $y_j \in [x_{j-1}, x_j]$

We set $\mathcal{R}_{\mathcal{P}, \{y_j\}_{j=1}^n}(f) = \sum_{j=1}^{n} (x_j - x_{j-1}) f(y_j)$

We call this a "Riemann sum" for f.



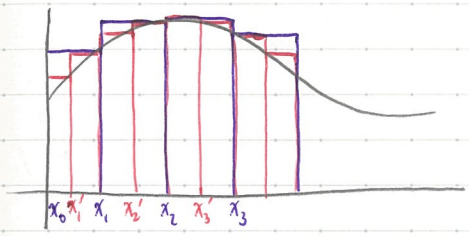
Remark

In each subinterval, we can choose y_j such that $f(y_j)$ takes the maximum value in the interval $[x_{j-1}, x_j]$, or choose y_j such that $f(y_j)$ takes the minimum value in $[x_{j-1}, x_j]$

The first Riemann sum is called $\mathcal{R}_P^{max}(f)$, and the second one is $\mathcal{R}_P^{min}(f)$.

We call them the upper/lower Riemann sum. One has

$$\mathcal{R}_P^{min}(f) \leq \mathcal{R}_{P,\{y_j\}_{j=1}^n}(f) \leq \mathcal{R}_P^{max}(f)$$



Idea: Consider lower & upper partition of $[a,b]$

- Consider a partition $P = \{x_j\}_{j=0}^n$, and a finer partition $P' = \{x_0, x_1', x_1, x_2', x_2, \dots, x_n\}$

Then $\mathcal{R}_P^{max}(f) \geqslant \mathcal{R}_{P'}^{max}(f)$ thinner → decreases

$\mathcal{R}_P^{min}(f) \leq \mathcal{R}_{P'}^{min}(f)$ thinner → increases

By taking finer partitions, $\mathcal{R}^{max}(f)$ is decreasing, while $\mathcal{R}^{min}(f)$ is increasing.

But $\mathcal{R}^{max}(f)$ is lower bounded (by $\mathcal{R}_P^{min}(f)$) and $\mathcal{R}^{min}(f)$ is upper bounded by $\mathcal{R}^{max}(f)$.

This implies that, by taking linear partition, the upper Riemann sums are converging, and the lower Riemann sums are also converging.

Def A bounded function $f: [a,b] \to \mathbb{R}$ is Riemann integrable if

supremum "maximum" infimum "minimum"

$$\sup_{P \text{ any partition of } [a,b]} \mathcal{R}_P^{min}(f) = \inf_{P \text{ partion of } [a,b]} \mathcal{R}_P^{max}(f)$$

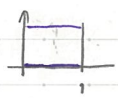
In this case, we write

$\int_a^b f(x)\,dx$ for this value.

just a notation for this number.

Example

Consider $[a,b] = [0,1]$



and $f(x) = \begin{cases} 1 & \text{if } x \in \mathbb{Q} \cap [0,1] \\ 0 & \text{if } x \in \mathbb{R} \setminus \mathbb{Q} \cap [0,1] \end{cases}$

Take any partition P of $[0,1]$

$\mathcal{R}_P^{min}(f) = 0$ because there are $y \in \mathbb{R} \setminus \mathbb{Q}$ in any subinterval.

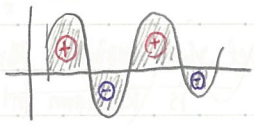
$$\mathcal{R}_P^{max}(f) = \sum_{j=1}^n (x_j - x_{j-1}) \cdot 1 = 1$$

$(x_1 - \cancel{x_0}^{0}) + (x_2 - x_1) + \dots + (x_n - x_{n-1})$

$\Rightarrow f$ is not Riemann integrable.

Thm Let $f: [a,b] \to \mathbb{R}$ be continuous, then f is Riemann integrable on $[a,b]$. (It means $\int_a^b f(x)dx$ is well defined).

Remark: If f is positive and continuous, the construction with Riemann sums give a precise meaning to the surface below the curve. But it is more general since f has not to be positive.



Properties of this integral

If f is Riemann integrable on $[a,b]$ and if $c \in (a,b)$, then

$$\int_a^b f(x)dx = \int_a^c f(x)dx + \int_c^b f(x)dx .$$

If $a<b$, we set (already defined)

$$\int_b^a f(x)dx := -\int_a^b f(x)dx$$

Then we can write

$$\int_x^z f(s)ds = \int_x^y f(s)ds + \int_y^z f(s)ds$$

for any x, y, z.

Thm (Extremion of previous thm.)
Let $f: [a,b] \to \mathbb{R}$ be continuous ($\Rightarrow$ Riemann integrable) and for $x \in [a,b]$ we set $F(x) = \int_a^x f(y)dy$ (limit of Riemann sum)

Then $F: (a,b) \to \mathbb{R}$ is differentiable, and $F' = f$

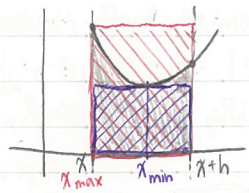
Proof: Consider $x \in (a,b)$ and $h>0$ such that $x+h \in (a,b)$ (The proof for $h<0$ is similar)

Then $$\frac{F(x+h)-F(x)}{h} = \frac{\int_a^{x+h} f(y)dy - \int_a^x f(y)dy}{h}$$

$$= \frac{\int_h^{x+h} f(y)dy}{h}$$

continue next class

(continue)



Let $x_{min} \in [x, x+h]$ be the global minimum of f on $[x, x+h]$, and x_{max} its global maximum. Then $f(x_{min}) \le f(y) \le f(x_{max})$ $\forall y \in [x, x+h]$

$$\Rightarrow f(x_{min})h \le \int_x^{x+h} f(y)dy \le h f(x_{max})$$

$$\Rightarrow f(x_{min}) \le \frac{\int_x^{x+h} f(y)dy}{h} \le f(x_{max})$$

Since $\lim_{h\to 0} x_{max} = x = \lim_{h\to 0} x_{min}$

One gets that

$$f(x) \le \lim_{h\to 0} \frac{\int_x^{x+h} f(y)dy}{h} \le f(x)$$

$\Rightarrow$ F is differentiable, and $F' = f$ #

The equality

$$\frac{d}{dx}\int_a^x f(y)dy = f(x)$$

called the Fundamental Theorem of Calculus.

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Reminder

- Indefinite integral for f
- Riemann sums
- $f: [a,b] \to \mathbb{R}$ Riemann integrable if
$$\lim_{\mathcal{P}} R_{min}(f, \mathcal{P}) = \lim_{\mathcal{P}} R_{max}(f, \mathcal{P})$$

VIII.4) Properties of the Integral

Let $f: [a,b] \to \mathbb{R}$ be continuous, let $x \in [a,b]$ and set $F(x) = \int_a^x f(y)\,dy$.
↑ is an indefinite integral for f

Let $G: [a,b] \to \mathbb{R}$ be another indefinite integral for $f \Rightarrow G = F + c$, with $c \in \mathbb{R}$.

Since $F(a) = 0 \Rightarrow G(a) = F(a) + c = c$

Then $\int_a^b f(y)\,dy = F(b) - F(a)$
$= G(b) - G(a) = G(x)\Big|_a^b$

Lemma: If $f, g: [a,b] \to \mathbb{R}$, continuous, then

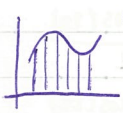
1.) $\int_a^b (f(y) + \lambda g(y))\,dy$
$= \int_a^b f(y)\,dy + \lambda \int_a^b g(y)\,dy$

2.) ⚠ $\int_a^b f(y)g(y)\,dy$
$\neq \int_a^b f(y)\,dy \cdot \int_a^b g(y)\,dy$

3.) If $f(x) \leq g(x) \Rightarrow \int_a^b f(y)\,dy \leq \int_a^b g(y)\,dy$

Proof for 3.

$\Rightarrow g(x) - f(x) \geq 0 \Rightarrow \int_a^b (g(x) - f(x))\,dx$

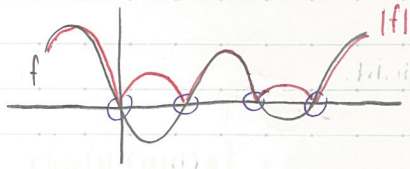


integral of positive function

$\Rightarrow \int_a^b g(x)\,dx - \int_a^b f(x)\,dx \geq 0$

Proposition Let $f: [a,b] \to \mathbb{R}$ be continuous

$$\left|\int_a^b f(x)\,dx\right| \leq \int_a^b |f(x)|\,dx$$



Proof Observe that $|f|$ is also continuous, which means that RHS is well-defined

Since $|f(x)| \geq f(x)$

③ $\Rightarrow \int_a^b f(x)\,dx \leq \int_a^b |f(x)|\,dx$

Observe also that $-|f(x)| \leq f(x)$

$\Rightarrow \int_a^b -|f(x)|\,dx \leq \int_a^b f(x)\,dx$

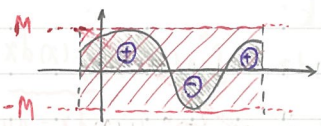
$\Rightarrow -\int_a^b |f(x)|\,dx \leq \int_a^b f(x)\,dx$

$\Rightarrow$ statement. #

Corollary

If $f: [a,b] \to \mathbb{R}$ is continuous and $|f(x)| \leq M < \infty \quad \forall x \in [a,b]$

then $\left|\int_a^b f(x)\,dx\right| \leq M(b-a)$



If $f(x) \leq M \Rightarrow \int_a^b f(x)\,dx \leq (b-a)M$

If $|f(x)| \leq M \Rightarrow \left|\int_a^b f(x)\,dx\right| \leq (b-a)M$

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IX.) Techniques of Integration

A few tricks.

IX.1) Substitution

Look for $\int f(x)\,dx = \int g(u(x))\,u'(x)\,dx$

$= G(u(x))$

with G an indefinite integral for g.

(Fund. Thm. of calculus)

Indeed $\frac{d}{dx}\int_a^x g(u(y))u'(y)\,dy = g(u(x))\,u'(x)$

and $\frac{d}{dx}G(u(x)) = g(u(x))\,u'(x)$

Examples

1) $\int \underbrace{(x^3+x)}_{u(x)}{}^9 \underbrace{(3x^2+1)}_{u'(x)}\,dx$ $\quad g(x) = x^9$

$= \frac{1}{10}(x^3+x)^{10} + \text{const.}$

2) $\frac{1}{2}\int \underbrace{2x}_{u'(x)} \sin(\underbrace{x^2}_{u(x)})\,dx$

$= -\cos(x^2) + \text{const.}$

Convenient trick:

$\int_a^b g(u(x))\,u'(x)\,dx = \int_a^b g(u(x))\,\frac{du}{dx}(x)\,dx$

$G(u(x))\Big|_{x=a}^{x=b}$ $\quad$ "$\Rightarrow$" $u'(x)\,dx = du$

$G(u(b)) - G(u(a)) = G(y)\Big|_{u(a)}^{u(b)}$

$= \int_{u(a)}^{u(b)} g(u)\,du$

Example

$\int x^5\sqrt{1-x^2}\,dx$

$= -\frac{1}{2}\int x^4\sqrt{1-x^2}\,(-2x)\,dx$

$= -\frac{1}{2}\int (1-u)^2 u^{\frac{1}{2}}\,du$

$= -\frac{1}{2}\int (u^{\frac{1}{2}} - 2u^{\frac{3}{2}} + u^{\frac{5}{2}})\,du$

$= -\frac{1}{2}\left(\frac{u^{\frac{3}{2}}}{3/2} - \frac{2}{5/2}u^{\frac{5}{2}} + \frac{1}{7/2}u^{\frac{7}{2}}\right) + \text{const.}$

$= -\frac{1}{3}(1-x^2)^{\frac{3}{2}} + \frac{2}{5}(1-x^2)^{\frac{5}{2}} - \frac{1}{7}(1-x^2)^{\frac{7}{2}} + \text{const.}$

IX.2) Integration by Parts.

Recall that $(fg)' = f'g + fg'$

$\Rightarrow$ $\int fg' = \int (fg)' - \int f'g = fg - \int f'g$

indefinite integral

Examples

1.) $\int \underbrace{x}_{f(x)} \underbrace{e^x}_{g'(x)}\,dx = xe^x - \int e^x\,dx$

$= xe^x - e^x = e^x(x-1)$

2.) $\int 1 \cdot \underbrace{\ln(x)}_{f(x)}\,dx = x\ln(x) - \int x\,\frac{1}{x}\,dx$

$\uparrow g'(x)$ $\quad = x\ln(x) - x$

IX.3) Trigonometric Integrals

Remember

$\sin^2(x) + \cos^2(x) = 1$

$\sin^2(x) = \frac{1-\cos(2x)}{2}$

$\cos^2(x) = \frac{1+\cos(2x)}{2}$

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Examples

1.) $\int \sin^2(x)\,dx = \frac{1}{2}\int (1-\cos(2x))\,dx$

$= \frac{1}{2}x - \frac{1}{4}\sin(2x)$

2.) $\int \cos^3(x)\,dx = \int \cos^2(x)\cos(x)\,dx$

$= \int (1-\sin^2(x))\cos(x)\,dx$

$= \sin(x) - \frac{1}{3}\sin^3(x)$

IX.4) Partial fraction

Calculus Lecture

Partial Fraction

Examples

1) $\int \frac{1}{x-a}\,dx = \ln(x-a)$

2) $\int \frac{1}{(x-a)^n}\,dx = \int (x-a)^{-n}\,dx = \frac{1}{-n+1}(x-a)^{-n+1}$ $(n \neq 1)$

3) $\int \frac{1}{x^2+1}\,dx = \arctan(x)$

More generally, consider $\frac{f(x)}{g(x)}$ with f, g polynomials with degree $(f) <$ degree (g).

In such a case, one has to factor the denominator into terms of the form

$(x-\alpha_j)^n$ or $((x-\beta_j)^2+\gamma_j^2)^m$

as for example

$$\frac{2x+5}{(x^2+1)^2(x-3)} = \frac{c_1+c_2x}{(x^2+1)} + \frac{c_3+c_4x}{(x^2+1)^2} + \frac{c_5}{x-3}$$

with $c_1, \dots, c_5 \in \mathbb{R}$

integral: $\frac{c_1}{x^2+1}$ (arctan) + $\frac{c_2x}{x^2+1}$ + $\frac{c_3}{(x^2+1)^2}$??? + $\frac{c_4x}{(x^2+1)^2}$ + $\frac{c_5}{x-3}$

$$= \frac{c_1(x^2+1)(x-3) + c_2x(x^2+1)(x-3) + c_3(x-3) + c_4x(x-3) + c_5(x^2+1)^2}{(x^2+1)^2(x-3)}$$

$$= \frac{1}{(x^2+1)^2(x-3)}\Big(\underbrace{(c_2+c_5)}_{=0}x^4 + \underbrace{(c_1-3c_2)}_{=0}x^3 + \underbrace{(-3c_1+c_2+c_4+2c_5)}_{=0}x^2 + \underbrace{(c_1-3c_2+c_3-3c_4)}_{=2}x + \underbrace{(-3c_1-3c_3+c_5)}_{=5} \Big)$$

We find

$$\begin{cases} c_1 = -\frac{33}{100} & c_3 = -\frac{130}{100} & c_5 = \frac{11}{100} \\ c_2 = -\frac{11}{100} & c_4 = -\frac{110}{100} & \end{cases}$$

The integrals of four terms are simple, but what about $\int \frac{1}{(x^2+1)^2}\,dx$?

Consider $\int \underbrace{\frac{1}{(x^2+1)}}_{g(x)}\,\underbrace{dx}_{f(x)} = \underbrace{\frac{x}{x^2+1}}_{=\arctan(x)} + 2\int \frac{x^2+1-1}{(x^2+1)^2}\,dx$

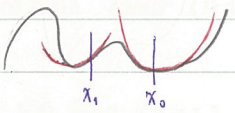
$$= \frac{x}{x^2+1} + \underbrace{2\int \frac{1}{x^2+1}\,dx}_{2\arctan(x)} - 2\int \frac{1}{(x^2+1)^2}\,dx$$

$$\int \frac{1}{(x^2+1)^2}\,dx = \frac{1}{2}\frac{x}{x^2+1} + \frac{1}{2}\arctan(x)$$

Remark: Similar tricks exist for $\int \frac{1}{(x^2+1)^m}\,dx$ for $m \in \mathbb{N}$.

X.) Taylor's Formula

Idea: approximate a function locally by a polynomial



Recall that for $f: [a,b] \to \mathbb{R}$, sufficiently differentiable, we write

$f^{(0)} = f$
$f^{(1)} = f'$
$f^{(2)} = f''$ (n times)
$f^{(n)} = f''''\dots' = f^{(n-1)'}$

Def Consider $f: [a,b] \to \mathbb{R}$, such differentiable, and let $x_0 \in (a,b)$. Let $n \in \mathbb{N}$

For any $x \in [a,b]$ we define $P_n(x_0, x)$

$$:= f(x_0) + f'(x_0)(x-x_0) + \frac{1}{2}f''(x_0)(x-x_0)^2 + \dots + \frac{1}{n!}f^{(n)}(x_0)(x-x_0)^n$$

$$= \sum_{j=0}^{n} \frac{1}{j!} f^{(j)}(x_0)(x-x_0)^j$$

$P_n(x_0, \cdot)$ is a polynomial of degree n, called the Taylor.

Question

$$\begin{aligned} f(x_0) - P_n(x_0, x_0) &= 0 \\ f'(x_0) - P_n'(x_0, x_0) &= 0 \\ f''(x_0) - P_n''(x_0, x_0) &= 0 \\ &\vdots \\ f^{(n)}(x_0) - P_n^{(n)}(x_0, x_0) &= 0 \end{aligned}$$

$f^{(j)}$ and $P_n^{(j)}(x_0, \cdot)$ take the same value at x_0 for any $j \in \{0, 1, 2, \dots, n\}$

What about $f(x) - P_n(x_0, x)$ for $x \neq x_0$?

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Example

1.) $f(x) = \sin(x)$, $x_0 = 0$, $n = 5$

$P_5(0,x) = x - \frac{1}{3}x^3 + \frac{1}{5}x^5$

What about $\sin(x) - x + \frac{1}{3!}x^3 - \frac{1}{5!}x^5$ for $x = 0$?

Thm (Taylor's expansion thm)

Let $f: [a,b] \to \mathbb{R}$. Let $n \in \mathbb{N}$ and suppose that f is $(n+1)$ times differentiable, with $f^{(n+1)}$ continuous ($\Leftrightarrow f \in C^{n+1}((a,b))$)

Let $x_0 \in (a,b)$ and $x \in [a,b]$

Then $f(x) = P_n(x_0, x) + R_{n+1}(x_0, x)$ (Remainder term)

with $R_{n+1}(x, x_0) = \int_{x_0}^{x} \frac{(x-t)^n}{n!} f^{(n+1)}(t)\, dt$

In addition,

$R_{n+1}(x, x_0) = \frac{1}{(n+1)!} f^{(n+1)}(c)(x-x_0)^{n+1}$

for some $c \in (x_0, x)$

Proof (by induction)

For $n=0$, one has

$f(x) \overset{?}{=} f(x_0) + R_1(x_0, x)$

$= \int_{x_0}^{x} f'(t)\, dt$

$\Leftrightarrow f(x) - f(x_0) = \int_{x_0}^{x} f'(t)\, dt$ ✓ Property of any indefinite integral

$\Rightarrow$ the statement is true for $n=0$

Assume now that the statement is true for a certain n, and let us prove it for $n+1$.

↳ $f(x) = P_n(x_0, x) + \int_{x_0}^{x} \frac{(x-t)^n}{n!} f^{(n+1)}(t)\, dt$

(int. by parts) $g'(t)$ $h(t)$

$\int_{x_0}^{x} g'(t)h(t)\, dt = g(t)h(t)\Big|_{t=x_0}^{t=x} - \int_{x_0}^{x} g(t)h'(t)\, dt$

$= -\frac{(x-t)^{n+1}}{(n+1)!} f^{(n+1)}(t)\Big|_{t=x_0}^{t=x} + \int_{x_0}^{x} \frac{(x-t)^{n+1}}{(n+1)!} f^{(n+2)}(t)\, dt$

$= \frac{(x-x_0)^{n+1}}{(n+1)!} f^{n+1}(x_0) + R_{n+2}(x, x_0)$

$\Rightarrow f(x) = P_n(x_0, x) + \frac{(x-x_0)^{n+1}}{(n+1)!} f^{(n+1)}(x_0) + R_{n+2}(x, x_0)$

$= P_{n+1}(x_0, x) + R_{n+2}(x, x_0)$

$\Rightarrow$ We have proved the result for $n+1$. Since n is arbitrary, the first statement is proved.

For the second statement, since $f^{(n+1)}$ is continuous, there exist x_{max} and x_{min} in $[x_0, x]$ with $f^{(n+1)}(x_{min}) \le f^{(n+1)}(t) \le f^{(n+1)}(x_{max})$ $\forall t \in [x_0, x]$.

Suppose that $x_0 \le x$, then

Then $\int_{x_0}^{x} \frac{(x-t)^n}{n!} f^{(n+1)}(x_{min})\, dt \le \int_{x_0}^{x} \frac{(x-t)^n}{n!} f^{(n+1)}(t)\, dt \le \int_{x_0}^{x} \frac{(x-t)^n}{n!} f^{(n+1)}(x_{max})\, dt$

$\frac{(x-x_0)^{n+1}}{(n+1)!} f^{(n+1)}(x_{min}) \le R_{n+1}(x, x_0) \le \frac{(x-x_0)^{n+1}}{(n+1)!} f^{(n+1)}(x_{max})$

$\Rightarrow \exists c \in [x_0, x]$ with $f^{(n+1)}(x_{min}) \le f^{(n+1)}(c) \le f^{(n+1)}(x_{max})$

with $f^{(n+1)}(c) \cdot \frac{(x-x_0)^{(n+1)}}{(n+1)!} = R_{n+1}(x, x_0)$

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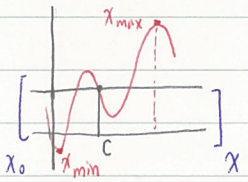

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Reminder

- For $f: \mathbb{R} \to \mathbb{R}$ differentiable many times, for fixed $x_0 \in \mathbb{R}$, and $n \in \mathbb{N}$, set

$$P_n(x_0, x) := \sum_{j=0}^{n} \frac{1}{j!} f^{(j)}(x_0)(x - x_0)^j$$

↑ polynomial of degree n

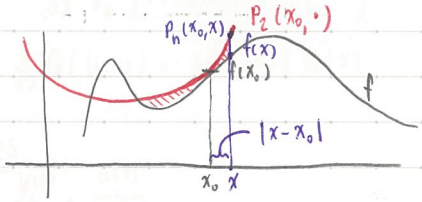
- Taylor: $f(x) = P_n(x_0, x) + R_{n+1}(x_0, x)$

with $R_{n+1}(x_0, x)$

$$= \int_{x_0}^{x} \frac{(x-t)^n}{n!} f^{(n+1)}(t)\, dt$$

$$= \frac{1}{(n+1)!} f^{(n+1)}(c)(x - x_0)^{n+1}$$

for some $c \in (x_0, x)$

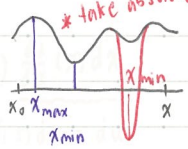


$$|R_{n+1}(x_0, x)| = \left| \frac{1}{(n+1)!} f^{(n+1)}(c)(x - x_0)^{n+1} \right|$$

$$= \frac{1}{(n+1)!} |x - x_0|^{n+1} |f^{(n+1)}(c)|$$

$$\leq \frac{1}{(n+1)!} |x - x_0|^{n+1} \sup_{c \in [x, x_0]} |f^{(n+1)}(c)|$$

$$= \frac{1}{(n+1)!} |x - x_0|^{n+1} \max\left\{ |f^{(n+1)}(x_{max})|, |f^{(n+1)}(x_{min})| \right\}$$



Examples

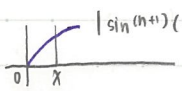
1.) $f(x) = \sin(x)$

$x_0 = 0$, n fixed

$$f(x) = x - \frac{x^3}{3!} + \frac{x^5}{5!} - \frac{x^7}{7!} + \ldots + R_{n+1}(0, x)$$

with $|R_{n+1}(0, x)| \leq \frac{1}{(n+1)!} |x|^{n+1} \cdot 1$

decreases as n increases



2) $f(x) = e^x$

$x_0 = 0$, $x > 0$

fixed $n \in \mathbb{N}$

$$f(x) = e^x = 1 + x + \frac{x^2}{2!} + \frac{x^3}{3!} + \ldots + R_{n+1}(0, x)$$

with $|R_{n+1}(0, x)| \leq \frac{1}{(n+1)!} |x|^{n+1} e^x$, $\leq e$ if $x \leq 1$

Lemma

For any $x \in \mathbb{R}$ one has

$$\lim_{n \to \infty} \frac{|x|^n}{n!} = 0$$

This implies that the remainder term is "usually" decreasing with n

$$R_{n+1}(0, x) = \frac{1}{(n+1)!} |x|^{n+1} f^{(n+1)}(c)$$

Proof

Consider $x > 0$ (or add absolute value) and let $n_0 \in \mathbb{N}$ such that

$$x < \frac{n_0}{2} \iff \frac{x}{n_0} < \frac{1}{2}$$

Then $\frac{x^n}{n!} = \frac{x}{1} \cdot \frac{x}{2} \cdot \frac{x}{3} \cdots \frac{x}{n_0} \cdot \frac{x}{(n_0+1)} \cdots \frac{x}{n}$ for $n > n_0$

($n_0 - 1$ term; $n - n_0 + 1$ term)

$$\leq \underbrace{\frac{x^{n_0-1}}{(n_0-1)!}}_{\text{indep. of } n \text{ (decreasing)}} \underbrace{\left(\frac{1}{2}\right)^{n-n_0+1}}_{\to 0 \text{ as } n \to \infty}$$

This goes to 0 as $n \to \infty$

#

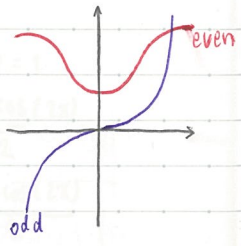
Odd vs. Even functions

Remember:

f is even if $f(-x) = f(x)$

f is odd if $f(-x) = -f(x)$

$\iff -f(-x) = f(x)$



Taylor expansion for odd/even functions.

- Recall that if $f: \mathbb{R} \to \mathbb{R}$ is differentiable,

$$f'(x) = \lim_{h \to 0} \frac{f(x+h) - f(x)}{h}$$

Question: If f is even, what about f' ?

Lemma Let f be differentiable.
If f is even, f' is odd, and
if f is odd, f' is even.

Proof Suppose f odd. One has

f is odd ((
$$\frac{f(x+h) - f(x)}{h} \xrightarrow{h \to 0} f'(x)$$
$$= \frac{-f(-(x+h)) - (-f(-x))}{h} \quad f' \text{ is even}$$
$$= \frac{f(-x-h) - f(-x)}{h} \xrightarrow{(-)h \to 0} f'(-x)$$

If f is even,
$$\frac{f(x+h) - f(x)}{h} \xrightarrow{h \to 0} f'(x) \quad f' \text{ is odd}$$

f is even ((
$$-\frac{f(-x-h) - f(-x)}{-h} \xrightarrow{h \to 0} -f'(-x)$$

Remark
If f is odd, $f(0) = 0$
(since $f(-x) = -f(x)$, take $x = 0$)

Thm Let f be sufficiently differentiable, and consider $P_n(0, x)$ its Talor polynomial at 0

$$\sum_{j=0}^{n} \frac{1}{j!} f^{(j)}(0)(x)^j$$

If f is even, then $f^{(j)}(0) = 0$ whenever j is odd.
If f is odd, then $f^{(j)}(0) = 0$ whenever j is even.

Proof
If f is even, f' is odd, f'' is even, f''' is odd
... $f^{(j)}$ is odd whenever j is odd.
$\Rightarrow$ $f^{(j)}(0) = 0$ "————" (by the remark)
If f is odd, $f^{(j)}$ is odd whenever j is even.
$\Rightarrow$ $f^{(j)} = 0$ "————" #

$$f(x) = P_n(x_0, x) + R_{n+1}(x_0, x)$$

Question: Can we take $n \to \infty$?

motivation for next chapter !

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XI.) Series

XI.1) Convergent series. (indefinite sums)

Consider $a_1, a_2, a_3, a_4, \dots$ be a sequence of real numbers.

When is $\sum_{j=1}^{\infty} a_j$ finite? or convergent?

For example, consider $a_j = \begin{cases} 1 & \text{if } j \text{ is even} \\ -1 & \text{if } j \text{ is odd} \end{cases}$

What is $-1+1-1+1-1+1-1+1\dots$

$(-1+1) + (-1+1) + (-1+1) + \dots = 0$

$-1 + (1-1) + (1-1) + (1-1) + \dots = -1$ etc.

For any sequence $a_1, a_2, a_3, \dots =: (a_j)_{j=1}^{\infty}$ we set $S_n = \sum_{j=1}^{n} a_j = a_1 + a_2 + a_3 + \dots + a_n$

S_n: partial sum

Def The series $a_1 + a_2 + a_3 + \dots$ is convergent if $\lim_{n\to\infty} S_n$ exists.

If the limit of S_n does not exist when $n \to \infty$, we say that the series is not convergent, or that it is divergent.

If the sequence S_n is convergent, then we write $\lim_{n\to\infty} S_n = \sum_{j=1}^{\infty} a_j$.

Examples

1.) Consider $a_j = \frac{1}{2^{j-1}}$

The series is $1 + \frac{1}{2} + \frac{1}{4} + \frac{1}{8} + \frac{1}{16} + \dots$

(called the geometric series)

In this case, $S_n = a_1 + a_2 + a_3 + \dots + a_n$

$$= \sum_{j=1}^{n} \frac{1}{2^{j-1}} = \sum_{j=1}^{n} \left(\frac{1}{2}\right)^{j-1} = \sum_{k=0}^{n-1} \left(\frac{1}{2}\right)^{k}$$

$$= \frac{1-\left(\frac{1}{2}\right)^n}{1/2} = 2\left(1-\left(\frac{1}{2}\right)^n\right) \xrightarrow{n\to\infty} 2$$

It means that the partial sum S_n converge to 2, which means that the series is convergent. $\Rightarrow \sum_{j=1}^{\infty} a_j = 2$

2) $a_j = \begin{cases} 1 & \text{if } j \text{ is even} \\ -1 & \text{if } j \text{ is odd} \end{cases}$

Then $S_n = \begin{cases} -1 & \text{if } n \text{ is odd} \\ 0 & \text{if } n \text{ is even} \end{cases}$

The partial sums are not converging, so the series does not converge.

3) $a_j = \frac{1}{j}$

$S_n = 1 + \frac{1}{2} + \frac{1}{3} + \frac{1}{4} + \frac{1}{5} + \frac{1}{6} + \frac{1}{7} + \frac{1}{8} + \dots + \frac{1}{16} + \dots + \frac{1}{n}$

$1 + \frac{1}{2} > 2 \cdot \frac{1}{4} = \frac{1}{2}$ $> 4 \cdot \frac{1}{8} = \frac{1}{2}$ $> 8 \cdot \frac{1}{16} = \frac{1}{2}$ $\dots \sim \ln(n)$

$\xrightarrow{n\to\infty} \infty$

The series does not converge.

$$\sum_{j=0}^{n} a^j = \frac{1-a^{n+1}}{1-a}$$

$\sum a_j x^j$

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Reminder:

- Taylor's formula for even / odd functions
- Sequence of number (ordered) $(a_j)_{j=1}^{\infty} \neq \{a_j\}_{j=1}^{\infty}$ (not ordered)
  $\neq$ series of number $a_1 + a_2 + a_3 + \dots$
- Partial sums and convergent series

Lemma Let $(a_j)_{j=1}^{\infty}$, $(b_j)_{j=1}^{\infty}$ be sequence of real numbers with convergent associated series.

Then

1.) $\sum_{j=1}^{\infty} (\lambda a_j) = \lambda \sum_j a_j \Rightarrow$ This series is convergent

2.) $\sum_j (a_j + b_j) = \sum_j a_j + \sum_j b_j$

3.) $(\sum_{j=1}^{n} a_j)(\sum_{k=1}^{n} b_k)$ this is partial sum converging to $(\sum_{j=1}^{\infty} a_j)(\sum_{k=1}^{\infty} b_k)$

4.) $\sum_{j=1}^{n} a_j b_j \neq (\sum_{j=1}^{n} a_j)(\sum_{j=1}^{n} b_j)$ in general

2) Series with positive terms only

Remark When all $a_j \geq 0$, the sequence of partial sum $S_n = \sum_{j=1}^{n} a_j$ is an increasing sequence

Thm (monotone convergence theorem)
An increasing sequence $(S_n)_{n=1}^{\infty}$ of real numbers which is upper bounded is convergent

Corollary A series of positive numbers is convergent if the sequence of partial sums is upper bounded.

Examples

$$a_j = \frac{1}{j^2}$$

We consider the series

$$1 + \underbrace{\frac{1}{2^2} + \frac{1}{3^2}}_{\leq 2 \cdot \frac{1}{2^2}} + \underbrace{\frac{1}{4^2} + \frac{1}{5^2} + \frac{1}{6^2} + \frac{1}{7^2}}_{\leq 4 \cdot \frac{1}{4^2}} + \underbrace{\frac{1}{8^2} + \frac{1}{9^2} + \dots}_{\leq 8 \cdot \frac{1}{8^2}}$$

$$= 1 + \frac{1}{2} + \frac{1}{4} + \frac{1}{8} + \dots = \sum_{j=0}^{\infty} \frac{1}{2^j} = 2$$

$\Rightarrow \sum \frac{1}{j^2} < 2$ which implies that the series is convergent, in fact $\sum_{j=1}^{\infty} \frac{1}{j^2} = \frac{\pi^2}{6}$ Basel Problem

Corollary (Comparison lemma)
Let $(a_j)_j$, $(b_j)_j$ be sequences of positive numbers, and assume that the series $\sum_{j=1}^{\infty} b_j$ is convergent. If there exists $d > 0$ such that $a_j \leq d b_j \ \forall j$, then the series $\sum a_j$ is convergent, and $\sum_{j=1}^{\infty} a_j \leq d \sum_{j=1}^{\infty} b_j$

3) Absolute Convergence

In this section, we do not assume $a_j \geq 0$.

Def A series $\sum_{j=1}^{\infty} a_j$ is absolutely convergent if the series $\sum |a_j|$ is convergent.

Thm If a series is absolutely convergent, then it is convergent.

⚠ The converse is not true in general

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Counter example

The series $\sum_{j=1}^{\infty} (-1)^j \frac{1}{j} = -1 + \frac{1}{2} - \frac{1}{3} + \frac{1}{4} - \frac{1}{5} + \dots$

is converging, but

$$\sum_{j=1}^{\infty} \left| (-1)^j \frac{1}{j} \right| = \sum_{j=1}^{\infty} \frac{1}{j} = \infty$$

Proof of the thm

Consider the sequence $(a_j)_{j=1}^{\infty}$, and suppose that $\sum_{j=1}^{\infty} |a_j|$ is convergent. Set $b_j = a_j$ if $a_j \geq 0$ and $b_j = 0$ if $a_j < 0$. Set $c_j = -a_j$ if $a_j < 0$, and $c_j = 0$ if $a_j \geq 0$.

Observe that $a_j = b_j - c_j$ and $|a_j| = b_j + c_j$

Since $\sum_{j=1}^{\infty} |a_j| < \infty$, then $\sum_{j=1}^{\infty} (b_j + c_j) < \infty \Leftrightarrow \sum_j b_j < \infty$ and $\sum_j c_j < \infty$

Now, consider $S_n = \sum_{j=1}^{n} a_j = \sum_{j=1}^{n} b_j - \sum_{j=1}^{n} c_j$

Since $\lim_{n\to\infty} \sum_{j=1}^{n} b_j < \infty$ and $\lim_{n\to\infty} \sum_{j=1}^{n} c_j < \infty$

then $\lim_{n\to\infty} \sum_{j=1}^{n} a_j = \lim_{n\to\infty} \sum_{j=1}^{n} b_j - \lim_{n\to\infty} \sum_{j=1}^{n} c_j$

$= \sum_{j=1}^{n} b_j - \sum_{j=1}^{n} c_j$ which exists. #

⚠ When a series is not absolutely convergent, one has to manipulate it carefully.

Thm (Criterion for alternating series)

Let $(a_j)_{n=1}^{\infty}$ be a sequence of numbers such that

1.) $\lim_{j\to\infty} a_j = 0$

2.) $a_j a_{j+1} \leq 0 \quad \forall j$

3.) $|a_{j+1}| \leq |a_j| \quad \forall j$

Then the series $\sum_{j=1}^{\infty} a_j$ is converging.

Remark This thm implies that the series $-1 + \frac{1}{2} - \frac{1}{3} + \frac{1}{4} - \frac{1}{5} + \frac{1}{6} - \dots$ is converging.

XI.4) Power Series

Def A power series is an expression of the form $\sum_{j=0}^{\infty} a_j x^j$ with $a_j \in \mathbb{R}$ and x a variable.

Example

The power series $\sum_{j=0}^{\infty} \frac{1}{j!} x^j$ corresponds to the exponential function.

Thm Assume that there exists $r \geq 0$ such that $\sum_{j=0}^{\infty} |a_j| r^j$ is convergent. Then for any $x \in \mathbb{R}$ with $|x| \leq r$, the series $\sum_{j=0}^{\infty} a_j x^j$ is absolutely convergent.

Proof Since $|a_j x^j| = |a_j||x|^j \leq |a_j| r^j$ and since the series $\sum_{j=0}^{\infty} |a_j| r^j$ is convergent, we got by the comparison lemma that $\sum_{j=0}^{\infty} |a_j x^j|$ is convergent, which means that $\sum_{j=1}^{\infty} a_j x^j$ is absolutely convergent. #

Def: The least upper bound on $r \geq 0$ such that $\sum_{j=0}^{\infty} |a_j| r^j < \infty$ is called "the radius of convergence" of the power series $\sum a_j x^j$

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Reminder

- Power series $f(x) := \sum_{j=0}^{\infty} a_j x^j$

  example $a_j = \frac{1}{j!} \rightsquigarrow f(x) = e^x$

- **Thm**:

  If $\sum_{j=0}^{\infty} |a_j| r^j < \infty$ for some $r \geq 0$, then $\sum_{j=0}^{\infty} a_j x^j$ converges for all $|x| \leq r$

- Def: The least upper bound on $r \geq 0$ such that $\sum_j |a_j| r^j < \infty$ is called the "radius of convergence"

- * can be 0, finite or ∞

Example

1.) $a_j = 1$, $f(x) = \sum_{j=0}^{\infty} x^j$

For $|x| < 1$ one has $\sum_{j=0}^{\infty} x^j = \frac{1}{1-x}$

but for $x = 1$, one has $\sum_{j=0}^{\infty} 1^j = \infty$

$\Rightarrow$ The radius of convergence is 1

2.) $a_j = \frac{1}{j!}$, we consider $f(x) = \sum_{j=0}^{\infty} \frac{1}{j!} x^j$

Let us prove that the radius of convergence is ∞.

We consider $\sum_{j=0}^{\infty} \frac{1}{j!} r^j$ and use the ratio test to show that this series is convergent.

Ratio test:

$$\frac{\frac{1}{(j+1)!} r^{j+1}}{\frac{1}{j!} r^j} = \frac{j!}{(j+1)!} \frac{r^{j+1}}{r^j} = \frac{1}{j+1} r \leq C < 1 \quad \forall j \geq N$$

For a fixed $r \geq 0$. If we choose $N \geq r$, then $\frac{1}{j+1} r \leq \underbrace{\frac{1}{n+1} r}_{\text{fix} < C} < 1$

Conclusion: For any fixed r, we can find $N \geq r$ and $C := \frac{r}{r+1} < 1$ such that $\frac{1}{j+1} r \leq C < 1$ $\forall j > N$

$\Rightarrow$ The series $\sum_{j=0}^{\infty} \frac{1}{j!} r^j$ is convergent

Since r is arbitrary, the radius of convergence is ∞.

Question: Can we have $a_{j+1} > a_j$ and still a non-zero radius of convergence?

3) Consider $a_j = \ln(j)$, it means $f(x) = \sum_{j=1}^{\infty} \ln(j) x^j$

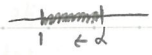
Let us consider $0 \leq r < 1$, and show that the series $\sum_{j=1}^{\infty} \ln(j) r^j < \infty$

By the ratio test we have

$$\frac{\ln(j+1) r^{j+1}}{\ln(j) r^j} = \underbrace{\frac{\ln(j+1)}{\ln(j)}}_{>1 \ ?} r \leq C < 1 \quad \text{for } j \text{ large enough.}$$

Observe that $\lim_{j \to \infty} \frac{\ln(j+1)}{\ln(j)} = 1$

We show that for any $\alpha > 1$



One has $\frac{\ln(j+1)}{\ln(j)} \leq \alpha$ for j large enough.

Indeed $\ln(j+1) \leq \alpha \ln(j) = \ln(j^\alpha)$

$\Leftrightarrow e^{\ln(j+1)} \leq e^{\ln(j^\alpha)}$

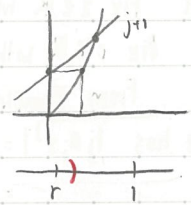
$\Leftrightarrow j+1 \leq j^\alpha$ which is true for j large enough.

Conclusion: $\lim_{j \to \infty} \frac{\ln(j+1)}{\ln(j)} = 1$

$\Rightarrow$ for any $r < 1$, we can find N such that

$\frac{\ln(j+1)}{\ln(j)} r \leq C < 1$ for $j \geq N$

very close to 1



$\Rightarrow \sum_{j=1}^{\infty} \ln(j) r^j$ is convergent for any $r < 1$

$\Rightarrow$ The radius of convergence in 1

Remark

The function $x \mapsto e^x$

$x \mapsto \sin(x)$

$x \mapsto \cos(x)$

can be expressed as power series with radius of convergence $= \infty$.

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Question: Suppose that the function

$$f(x) = \sum_{j=0}^{\infty} a_j x^j$$

has a radius of convergence equals to $R > 0$.
Can we differentiate f, and is $f' = \sum_{j=1}^{\infty} a_j j x^{j-1}$?

XI.5) Differentiation and Integration of power series.

Proposition: Consider $\sum_{j=0}^{\infty} a_j x^j$ and suppose that its radius of convergence is $R > 0$. Let $0 < r < R$.
(which means that $\sum_{j=0}^{\infty} a_j x^j$ is absolutely convergent for any $|x| \leq r$)

Then, the series $\sum_{j=1}^{\infty} j a_j x^{j-1}$ is also absolutely convergent for $|x| < r$.

Proof Fix $x \in \mathbb{R}$ with $x \neq 0$ and $|x| < r$ and let us fix $c \in \mathbb{R}$ with $|x| < c < r$.

From Homework 8, Ex. 3, one has $\lim_{n\to\infty} n^{\frac{1}{n}} = 1$

One has $|j a_j x^j| = |a_j (j^{1/j} x)^j|$

$= |a_j| |j^{\frac{1}{j}} x|^j \leq |a_j| c^j$

for j large enough.

$\Rightarrow |j a_j x^{j-1}| = \frac{1}{|x|} |a_j (j^{\frac{1}{j}} x)^j| \leq \frac{1}{|x|} |a_j| c^j$

Since the series $\sum_{j=0}^{\infty} a_j c^j$ is convergent, we get from the comparison lemma that $\sum_{j=1}^{\infty} j a_j x^{j-1}$ is absolutely convergent.

#

Theorem Let $\sum_{j=0}^{\infty} a_j x^j$ be a power series with a radius of convergence $R > 0$. Then the function $f: (-R, R) \ni x \mapsto \sum_{j=0}^{\infty} a_j x^j$ is differentiable, and $f'(x) = \sum_{j=1}^{\infty} j a_j x^{j-1}$.

Also, $\int f(x)dx = \sum_{j=0}^{\infty} \frac{1}{j+1} a_j x^{j+1} + \text{const.}$

Remark By applying the thm to f' (which is a power series with radius of convergence $R > 0$), we get that f' is differentiable and is given by $\sum_{j=2}^{\infty} j(j-1) a_j x^{j-2}$. This implies that f is C^∞.

Part of the Proof

We don't prove the first, but deduce the second part from the first statement.

Indeed, the series $\sum_{j=0}^{\infty} \frac{1}{j+1} a_j x^j$ is absolutely convergent for $|x| < R$ because $\frac{1}{j+1} |a_j| \leq |a_j|$ and the use of the comparison lemma, and thus the function

$$F(x) := \sum_{j=0}^{\infty} \frac{1}{j+1} a_j x^{j+1}$$

is differentiable (first part of the statement) and $F'(x) = \sum_{j=0}^{\infty} \frac{1}{j+1} a_j (j+1) x^j = f(x)$

#.

Remark: We have proved the initial manipulation on $\sum_{j=0}^{\infty} \frac{1}{j!} x^j$

Remark: Power series are very important for complex analysis.

The End ☺.