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Thm 1.2.18

Let $E \subseteq \mathbb{R}^n$. If E is L.m. (that is, Lebesgue measurable), then E^c is L.m.

<proof>

First, we admit the properties introduced in the lecture.
e.g. If $m^*(V)=0$, then V is L.m.
Another example : If V is closed, then V is L.m. (Thm 1.2.17).
Notice that we never use this Thm 1.2.18 to prove those properties.

Since E is L.m., for any $k \in \mathbb{N}$ $\exists G_k$: open s.t.

$$E \subset G_k \subset \mathbb{R}^n$$
$$m^*(G_k \setminus E) < \frac{1}{k}.$$

Since for $k \in \mathbb{N}$ $G_k^c \subseteq E^c$, we have $\bigcup_{k\in\mathbb{N}} G_k^c \subseteq E^c$.

Set $Z = E^c \setminus \bigcup_{k\in\mathbb{N}} G_k^c$

Then $E^c = Z \cup \bigcup_{k\in\mathbb{N}} G_k^c$.

Since G_k^c is closed, G_k^c is L.m. Thus $\bigcup_{k\in\mathbb{N}} G_k^c$ is also L.m.

ETS Z is L.m.

WTS $m^*(Z)=0$.

WTS $Z \subset G_k \setminus E$ for $k \in \mathbb{N}$

Let $z \in Z$ be arbitrary.
Since $z \in E^c \setminus \bigcup_{k\in\mathbb{N}} G_k^c$,
$z \notin E$ and $z \in \left(\bigcup_{k\in\mathbb{N}} G_k^c\right)^c = \bigcap_{k\in\mathbb{N}} G_k$

Thus $z \in G_k \setminus E$

We have $Z \subset G_k \setminus E$ and so

$$m^*(Z) \le m^*(G_k \setminus E) < \frac{1}{k}.$$

Since $k \in \mathbb{N}$ is arbitrary, $m^*(Z)$ must be zero. Thus Z is L.m.
Hence E^c is L.m. ∎

ETS = enough to show

WTS = want to show

Proposition

(※ This prop. is not from the lectures, but it might be useful.)

Let $f : [a, b] \to \mathbb{R}$.

TFAE

(1) f is Lebesgue measurable.

(2) $\forall t \in \mathbb{R}$, $f^{-1}[(-\infty, t)]$ is Lebesgue measurable.

(3) $\forall U \subset \mathbb{R}$: open, $f^{-1}[U]$ is Lebesgue measurable.

<proof>

[(3) ⇒ (1)] Since $(s, +\infty)$ is open for any $s \in \mathbb{R}$, $f^{-1}[(s, +\infty)]$ is L.m. (= Lebesgue measurable.)

[(1) ⇔ (2)]

[⇒] Let $t \in \mathbb{R}$ be arbitrary.

$$(-\infty, t)^c = [t, +\infty)$$

$$= \bigcap_{n \in \mathbb{N}} \left(t - \frac{1}{n}, +\infty\right)$$

Indeed, since for any $n \in \mathbb{N}$ $[t, +\infty) \subset \left(t - \frac{1}{n}, +\infty\right)$, we have $[t, +\infty) \subset \bigcap_{n \in \mathbb{N}} \left(t - \frac{1}{n}, +\infty\right)$.

For any $x \in$ RHS, if $x < t$, then $\exists n_0 \in \mathbb{N}$ s.t. $x < t - \frac{1}{n_0}$.

Thus $x \geq t$. Hence $x \in$ LHS.

Thus

$$\left(f^{-1}[(-\infty, t)]\right)^c = f^{-1}[(-\infty, t)^c]$$

$$= \bigcap_{n \in \mathbb{N}} f^{-1}\left[\left(t - \frac{1}{n}, +\infty\right)\right]$$

Since $f^{-1}\left[\left(t - \frac{1}{n}, +\infty\right)\right]$ is L.m., $\bigcap_{n \in \mathbb{N}} f^{-1}\left[\left(t - \frac{1}{n}, +\infty\right)\right]$ is L.m.

Thus $\left(f^{-1}[(-\infty, t)]\right)^c$ is L.m.

By Thm 1.2.18, $f^{-1}[(-\infty, t)]$ is L.m.

[⇐] Let $s \in \mathbb{R}$ be arbitrary.

$$(s, +\infty)^c = (-\infty, s] = \bigcap_{n \in \mathbb{N}} \left(-\infty, s + \frac{1}{n}\right)$$

Thus $\left(f^{-1}[(s, +\infty)]\right)^c = f^{-1}[(s, +\infty)^c]$

$$= \bigcap_{n \in \mathbb{N}} f^{-1}\left[\left(-\infty, s + \frac{1}{n}\right)\right]$$

Hence $f^{-1}[(s, +\infty)]$ is L.m. similarly to [⇒].

[(1) ⇒ (3)]

Since (1) ⇔ (2), we can use (2).

Let $U \subset \mathbb{R}$: open be arbitrary.

For any $x \in U$, $\exists \varepsilon > 0$ s.t. $B(x, \varepsilon) \subset U$.

Since $\mathbb{Q}$ is dense, $\exists (p_x, q_x) \in \mathbb{Q}^2$ s.t. $x \in (p_x, q_x) \subset B(x, \varepsilon)$.

Then $U = \bigcup_{x \in U} (p_x, q_x)$

Consider $\phi : U \to \mathbb{Q}^2$, $x \mapsto (p_x, q_x)$.

Let $\sim$ be a relation on U defined by $x \sim y \overset{\text{def}}{\Leftrightarrow} (p_x, q_x) = (p_y, q_y)$.

Then $\sim$ is an equivalence relation and $\bar{\phi}(\bar{x}) : U/\sim \to \mathbb{Q}^2$, $\bar{x} \mapsto (p_x, q_x)$ is well-defined and injective.

Indeed, $\bar{x} = \bar{y} \Leftrightarrow (p_x, q_x) = (p_y, q_y)$

$\Leftrightarrow \bar{\phi}(\bar{x}) = \bar{\phi}(\bar{y})$

Also, $U = \bigcup_{x \in U} (p_x, q_x) = \bigcup_{\bar{x} \in U/\sim} \bar{\phi}(\bar{x})$.

Since $\bar{\phi}$ is injective,
$|U/\sim| \leq |\mathbb{Q}^2| = |\mathbb{N}|$.

Thus, U is a finite or countably infinite union of sets that has the form (p_x, q_x).

Hence $f^{-1}[U]$ is a finite or countably infinite union of sets that has the form $f^{-1}[(p_x, q_x)]$, where $f^{-1}[(p_x, q_x)]$
$= f^{-1}[(p_x, +\infty)] \cap f^{-1}[(-\infty, q_x)]$, which is L.m.

Thus, $f^{-1}[U]$ is L.m.

■

Prop.

Let $f, g : [a,b] \to \mathbb{R}$ with $f = g$ a.e.
Then if f is L.m., then g is also L.m.

<proof>.
Let $N = \{x \in [a,b] \mid f(x) \neq g(x)\}$.
Since $f = g$ a.e., N is L.m. and $m(N) = 0$.

WTS $\forall s \in \mathbb{R}$, $g^{-1}[(s, +\infty)]$ is L.m.

$$g^{-1}[(s,+\infty)] = (g^{-1}[(s,+\infty)] \cap N) \cup (g^{-1}[(s,+\infty)] \cap N^c).$$

Let $A = g^{-1}[(s,+\infty)] \cap N$
Let $B = g^{-1}[(s,+\infty)] \cap N^c$.

Since $A \subset N$, we have $m^*(A) \leq m^*(N) = 0$.
Since $m^*(A) = 0$, A is L.m.

Since N is L.m., N^c is also L.m. (Thm 1.2.18).

$\forall x \in B$, $g^{-1}[(s,+\infty)] = f^{-1}[(s,+\infty)]$ since $g(x) = f(x)$.
Since f is L.m., $f^{-1}[(s,+\infty)]$ is L.m.
Thus $B = f^{-1}[(s,+\infty)] \cap N^c$ is L.m.
Hence $g^{-1}[(s,+\infty)] = A \cup B$ is L.m.
Thus g is L.m.

■

Thm

If each f_n is L.m. on $[a,b]$ and $\{f_n\}_{n\in\mathbb{N}}$ is pointwise bounded, then $f^*:[a,b]\to\mathbb{R}$

$f_*:[a,b]\to\mathbb{R}$

are L.m. where

$$f^*(x)=\lim_{n\to\infty}\sup_{m\geq n}\{f_m(x)\}=\inf_{n\geq 1}\sup_{m\geq n}\{f_m(x)\}.$$

$$f_*(x)=\lim_{n\to\infty}\inf_{m\geq n}\{f_m(x)\}=\sup_{n\geq 1}\inf_{m\geq n}\{f_m(x)\}$$

To prove this Thm, we prepare another proposition.

Prop

$\{f_n\}_{n\in\mathbb{N}}$: pointwise bounded

f_n: L.m.

Let $N\in\mathbb{N}$ and let

$$\begin{array}{rccc} \widehat{f_N}: & [a,b] & \longrightarrow & \mathbb{R} \\ & \Psi & & \Psi \\ & x & \longmapsto & \sup_{n\geq N}\{f_n(x)\} \end{array}$$

$$\begin{array}{rccc} \widehat{g_N}: & [a,b] & \longrightarrow & \mathbb{R} \\ & \Psi & & \\ & x & \longmapsto & \inf_{n\geq N}\{f_n(x)\}. \end{array}$$

Then $\{\widehat{f_N}\}$, $\{\widehat{g_N}\}$ are pointwise bounded. and $\widehat{f_N}$, $\widehat{g_N}$ are L.m. for $N\in\mathbb{N}$.

<proof>

Notice that $\widehat{f_N}$, $\widehat{g_N}$ are well-defined since $\{f_n\}$ is pointwise bounded.

① WTS $\widehat{f_N}$ is L.m.

Let $s\in\mathbb{R}$ be arbitrary. Then

$$\widehat{f_N}^{-1}[(s,+\infty)]=\bigcup_{n\geq N} f_n^{-1}[(s,+\infty)] \text{ —(*)}$$

Indeed,

$[\subseteq]$

If $x\in LHS$, then $\sup_{n\geq N}\{f_n(x)\}\in(s,+\infty)$.

Since s is not an upper bound of the sequence $\{f_n(x)\}_{n=N}^{\infty}$, $\exists n_0\in\mathbb{N}$ s.t.

$$s<f_{n_0}(x).$$

Thus $x\in RHS$.

$[\supseteq]$

If $x\in RHS$, $\exists n_0\in\mathbb{N}$ s.t.

$$s<f_{n_0}(x)\leq\sup_{n\geq N}\{f_n(x)\}.$$

Thus $x\in LHS$.

Since f_n is L.m., $\widehat{f_N}^{-1}[(s,+\infty)]$ is L.m. by (*).

② WTS $\widehat{g_N}$ is L.m.

ETS $\forall t\in\mathbb{R}$, $\widehat{g_N}^{-1}[(-\infty,t)]$ is L.m.

※ By the proposition on the page 2.

In the similar way as we saw in ①, we have $\widehat{g_N}^{-1}[(-\infty,t)]=\bigcup_{n\geq N} f_n^{-1}[(-\infty,t)]$

Since f_n is L.m., $\widehat{g_N}^{-1}[(-\infty,t)]$ is L.m. and so $\widehat{g_N}$ is L.m.

③ WTS $\widehat{f_N}$, $\widehat{g_N}$ are pointwise bounded.

Since $\{f_n\}$ is pointwise bounded, $\forall x\in\mathbb{R}$ $\exists m,M\in\mathbb{R}$, s.t. $m\leq f_n(x)\leq M$.

Thus $m\leq\inf_{n\geq N}\{f_n(x)\}\leq\sup_{n\geq N}\{f_n(x)\}\leq M$.

Hence $\widehat{f_N}$, $\widehat{g_N}$ are pointwise bounded ∎

To the backside

<Proof for the Thm>

$$f^*(x) = \inf_{N \geq 1} \sup_{n \geq N} \{f_n(x)\}$$
$$= \inf_{N \geq 1} \widetilde{f_N}(x).$$

Since $\{\widetilde{f_N}\}_{N \in \mathbb{N}}$ is pointwise bounded and $\widetilde{f_N}$ is L.m. by Prop,

f^* is L.m. by Prop

Likewise, f_* is L.m.

■

Cor

If $\{f_n\}$ is a sequence of L.m. function on $[a,b]$ and if $\exists f : [a,b] \to \mathbb{R}$ s.t.

$$\lim_{n \to \infty} f_n(x) = f(x) \text{ a.e.}$$

then f is L.m.

<proof>

Let $N = \{x \in [a,b] \mid \lim_{x \to \infty} f_n(x) \neq f(x)\}$.

Let
$$\begin{aligned} g : [a,b] &\to \mathbb{R} \\ \cup \quad & \quad \cup \\ x &\mapsto \lim_{n \to \infty} f_n(x) \end{aligned}$$

* If $\lim_{n \to \infty} f_n(x) \neq f(x)$, then we assign 0 to $g(x)$.

$M := \{x \in [a,b] \mid g(x) \neq f(x)\} \subset N$ implies

$m^*(M) \leq m^*(N) = 0$.

Thus M is L.m. and $m(M) = 0$.

Hence $g = f$ a.e. — (*)

For any $s \in \mathbb{R}$,

$$g^{-1}[(s,+\infty)] = \underbrace{(g^{-1}[(s,+\infty)] \cap N)}_{=:A} \cup \underbrace{(g^{-1}[(s,+\infty)] \cap N^c)}_{=:B}.$$

WTS A is L.m.

Since $A \subset N$, we have

$m^*(A) \leq m^*(N) = 0$.

WTS B is L.m.

Since $\lim_{n \to \infty} f_n(x)$ exists for $x \in B$,

we have

$\lim_{n \to \infty} f_n(x) = f^*(x)$ $(= f_*(x))$

Since f^* is L.m.,

$f^{*-1}[(s,+\infty)]$ is L.m.

Also, N^c is L.m.

Thus

$B = f^{*-1}[(s,+\infty)] \cap N^c$ is L.m.

Thus $g^{-1}[(s,+\infty)]$ is L.m.

Hence g is L.m.

By (*),

f is L.m.

■