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No. 1

Extension

For $n=3$, consider the function

$$h : \mathbb{R}^3 \setminus \{0\} \longrightarrow \mathbb{R}$$
$$x \longmapsto \frac{1}{\|x\|}$$

and observe that $h \in L^1_{loc}(\mathbb{R}^3)$.
Then $T_h \in \mathcal{D}'(\mathbb{R}^3)$.

<proof>

$L^1_{loc}(\mathbb{R}^3)$
$= \{ h : \mathbb{R}^3 \to \mathbb{R} \mid \int_{B(y,1)} |h(x)| dx < +\infty$ for any $y \in \mathbb{R}^3 \}$

We consider

$$h : \mathbb{R}^3 \longrightarrow \mathbb{R}$$
$$x \longmapsto \frac{1}{\|x\|}$$

with $h(0) = \infty$.

Let $y \in \mathbb{R}^3$ be arbitrary.
WTS
$$\int_{B(y,1)} |h(x)| dx < +\infty$$

To emphasize that this integral is Lebesgue, we use the following notation.

$$\int_{B(y,1)} |h(x)| dx = \int_{B(y,1)} |h| d\mu$$

where μ is Lebesgue measure.

(i) If $0 \notin \overline{B(y,1)}$, then

$$\int_{B(y,1)} |h| d\mu \leq \int_{\overline{B(y,1)}} h \, d\mu$$
$$\leq \sup_{x \in \overline{B(y,1)}} h(x) \, \mu\left(\overline{B(y,1)}\right)$$

Since h is continuous on $\overline{B(y,1)}$, which is compact, one has

$$\sup_{x \in \overline{B(y,1)}} h(x) < +\infty.$$

Thus, $\int_{B(y,1)} |h| d\mu < +\infty$.

(ii) If $0 \in \overline{B(y,1)}$, then $0 \in B(y,2)$.
Since $B(y,2)$ is open, $\exists \varepsilon > 0$ s.t.
$$B(0,\varepsilon) \subset B(y,2).$$

Then

$$\int_{B(y,1)} |h| d\mu$$
$$\leq \int_{B(y,2)} h \, d\mu$$
$$\leq \underbrace{\int_{B(0,\varepsilon)} h \, d\mu}_{I_1} + \underbrace{\int_{\overline{B(y,2) \setminus B(0,\varepsilon)}} h \, d\mu}_{I_2}$$

As with (i), we have $I_2 < +\infty$.

Set
$$h_n : \mathbb{R}^3 \longrightarrow \mathbb{R}$$
$$x \longmapsto \frac{1}{\|x\| + \frac{1}{n}}.$$

Since h_n is continuous, h_n is $\mathcal{B}(\mathbb{R}^3)$ measurable.
If $x \neq 0$, then
$$\lim_{n\to\infty} h_n(x) = \frac{1}{\|x\|} = h(x)$$
If $x = 0$, then
$$\lim_{n\to\infty} h_n(0) = \lim_{n\to\infty} n = \infty = h(0)$$

Thus $h_n \to h$ $(n \to \infty)$
Also, $h_n \geq 0$ and $h_n \leq h_{n+1}$ for $n \in \mathbb{N}$.
By the monotone convergence theorem,

$$I_1 = \lim_{n\to\infty} \int_{B(0,\varepsilon)} h_n \, d\mu$$

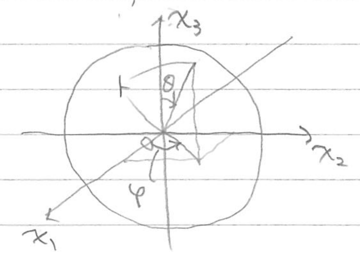
Set $I_n = \int_{B(0,\varepsilon)} h_n \, d\mu$

Since h_n is continuous,

$I_n = \int_{B(0,\varepsilon)} h_n(x)\,dx$ (Riemann)

$= \int_{B(0,\varepsilon)} \frac{1}{\|x\| + \frac{1}{n}} dx$

Using the spherical coordinate system, one has



$I_n = \int_0^\varepsilon dr \int_0^\pi d\theta \int_0^{2\pi} \frac{1}{r + \frac{1}{n}} \cdot r^2 \sin\theta \, d\varphi$

$= 2\pi \int_0^\varepsilon dr \, \frac{r^2}{r + \frac{1}{n}} \left[-\cos\theta\right]_0^\pi$

$= 4\pi \int_0^\varepsilon \frac{r^2}{r + \frac{1}{n}} dr$

Set $s = r + \frac{1}{n}$ Then

$I_n = 4\pi \int_{\frac{1}{n}}^{\frac{1}{n}+\varepsilon} \frac{(s - \frac{1}{n})^2}{s} ds$

$= 4\pi \int_{\frac{1}{n}}^{\frac{1}{n}+\varepsilon} \left(s - \frac{2}{n} + \frac{1}{n^2 s}\right) ds$

$= 4\pi \left[\frac{1}{2}s^2 - \frac{2}{n}s + \frac{1}{n^2}\log|s|\right]_{\frac{1}{n}}^{\frac{1}{n}+\varepsilon}$

$= 2\pi\left(\varepsilon^2 + \frac{2}{n}\varepsilon\right) - \frac{8\pi}{n} \cdot \varepsilon$

$+ \frac{1}{n^2} \log\left|\frac{\frac{1}{n}+\varepsilon}{\frac{1}{n}}\right|$

$= 2\pi\varepsilon^2 - \frac{4\pi\varepsilon}{n} + \frac{\log|n\varepsilon + 1|}{n^2}$

$\to 2\pi\varepsilon^2 \quad (n \to \infty)$

Thus, $I_1 = \lim_{n\to\infty} I_n = 2\pi\varepsilon^2 < +\infty$.

Hence,

$\int_{B(y,1)} |h| \, d\mu \leq I_1 + I_2 < +\infty$.

Thus, $h \in L^1_{loc}(\mathbb{R}^3)$

Hence, $T_h \in \mathcal{D}'(\mathbb{R}^3)$

<u>Remark</u>

If we assumed

$I_1 = \int_{B(0,\varepsilon)} h(x)\,dx$ (Riemann), then

$I_1 = \int_0^\varepsilon dr \int_0^\pi d\theta \int_0^{2\pi} \frac{1}{r} r^2 \sin\theta \, d\varphi$

$= 2\pi \int_0^\varepsilon dr \cdot r \left[-\cos\theta\right]_0^\pi$

$= 4\pi \int_0^\varepsilon r \, dr$

$= 4\pi \left[\frac{1}{2}r^2\right]_0^\varepsilon$

$= 2\pi\varepsilon^2$.