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Let $(A, D(A))$ be densely defined. Prove that : (a) $\ker(A^*) = \mathrm{Ran}(A)^\perp$

(b) If $A \subseteq B$ then $B^* \subseteq A^*$

Proof

(a) + If $f \in \ker(A^*)$ then $f \in D(A^*)$ and $A^*f = 0 \Rightarrow \langle f, Ag \rangle = \langle A^*f, g \rangle = 0 \ \forall g \in D(A) \Rightarrow f \in \mathrm{Ran}(A)^\perp$

+ Recall that $D(A^*) = \{ f \in \mathcal{H} \mid \exists f^* \in \mathcal{H} : \langle f^*, g \rangle = \langle f, Ag \rangle \}$ and $A^*f = f^* \ \forall f \in D(A^*)$

If $f \in \mathrm{Ran}(A)^\perp$ then $\langle f, Ag \rangle = 0 \ \forall g \in D(A)$. We have: $\langle f, Ag \rangle = 0 = \langle \underset{\in \mathcal{H}}{0}, g \rangle \ \forall g \in D(A)$

$\Rightarrow f \in D(A^*)$ and $A^*f = 0 \Rightarrow f \in \ker(A^*)$

$\Rightarrow \ker(A^*) = \mathrm{Ran}(A)^\perp$

(b) Let $f \in D(B^*) \Rightarrow \langle B^*f, g \rangle = \langle f, Bg \rangle \ \forall g \in D(B)$

Moreover, if $A \subseteq B$ then $\langle \underbrace{B^*f}_{\in \mathcal{H}}, g \rangle = \langle f, Ag \rangle \ \forall g \in D(A) \subseteq D(B) \Rightarrow f \in D(A^*)$ and $A^*f = B^*f$

$$\Rightarrow \begin{cases} D(B^*) \subseteq D(A^*) \\ A^*f = B^*f \quad \forall f \in D(B^*) \end{cases} \Rightarrow B^* \subseteq A^* \quad (A^* \text{ is an extension of } B^*)$$

Let $(A, D(A))$ be a self-adjoint operator ($D(A)$ is dense in $\mathcal{H}$)

Prove that if $B = B^* \in \mathcal{B}(\mathcal{H})$ then $(A+B, D(A))$ is a self-adjoint operator.

Proof:

$B \in \mathcal{B}(\mathcal{H})$ implies that $D(B) = \mathcal{H} \Rightarrow D(A+B) = D(A) \cap D(B) = D(A)$

+ If $f \in D((A+B)^*)$:

Suppose that $f^* \in \mathcal{H}$ such that $\langle f, (A+B)g \rangle = \langle f^*, g \rangle$; $f^* = (A+B)^*f$ with $f \in D((A+B)^*)$

$\Rightarrow \langle f^*, g \rangle = \langle f, (A+B)g \rangle = \langle f, Ag \rangle + \langle f, Bg \rangle = \langle f, Ag \rangle + \langle Bf, g \rangle \ \forall g \in D(A) \Rightarrow \langle f, Ag \rangle = \langle \underbrace{f^* - Bf}_{\in \mathcal{H}}, g \rangle$

$\Rightarrow f \in D(A^*) = D(A) \Rightarrow D((A+B)^*) \subseteq D(A) = D(A+B)$

and for $f \in D((A+B)^*) \subseteq D(A)$: $A^*f = f^* - Bf \Leftrightarrow Af = (A+B)^*f - Bf \Leftrightarrow (A+B)^*f = (A+B)f$

Thus, we have just obtained: $\begin{cases} D((A+B)^*) \subseteq D(A) = D(A+B) \\ (A+B)^*f = (A+B)f \quad \forall f \in D((A+B)^*) \end{cases}$ (1)

+ If $f \in D(A) = D(A+B)$

$\langle f, Ag \rangle = \langle f, (A+B)g \rangle - \langle f, Bg \rangle \Rightarrow \langle f, (A+B)g \rangle = \langle f, Ag \rangle + \langle f, Bg \rangle = \langle Af, g \rangle + \langle Bf, g \rangle = \langle \underbrace{(A+B)f}_{\in \mathcal{H}}, g \rangle$

$\Rightarrow f \in D((A+B)^*)$

$\Rightarrow D(A+B) \subseteq D((A+B)^*)$ (2)

From (1) and (2), one has $D((A+B)^*) = D(A+B)$ $(= D(A))$ and $(A+B)^*f = (A+B)f \quad \forall f \in D(A)$

$\Rightarrow (A+B)^* = (A+B) \Rightarrow (A+B, D(A+B))$, or equivalently $(A+B, D(A))$, is a self-adjoint operator.