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In $L^2(\mathbb{R}^n)$.

Recall $D_j := -i\partial_j$ and X_k multiplication by the variable x_k.

Then $\underline{[X_j, X_k] = 0}$ ① ; $\underline{[D_j, D_k] = 0}$ ② , $\underline{[iD_j, X_k] = \delta_{jk}}$ ③ on $S(\mathbb{R}^n)$.

Proof:

For $f \in S(\mathbb{R}^n)$:

① $[X_j, X_k] f = (X_j X_k - X_k X_j) f$

$= X_j (X_k f) - X_k (X_j f)$

$= X_j (x_k f) - X_k (x_j f)$

$= x_j x_k f - x_k x_j f$

$= 0 \cdot f \Rightarrow \underline{[X_j, X_k] = 0}$ $\square$

② $[D_j, D_k] f = (D_j D_k - D_k D_j) f$

$= D_j(-i\partial_k f) - D_k(-i\partial_j f)$

$= (-i\partial_j)(-i\partial_k f) - (-i\partial_k)(-i\partial_j f)$

$= -\partial_j(\partial_k f) + \partial_k(\partial_j f)$

$= 0 \cdot f \Rightarrow \underline{[D_j, D_k] = 0}$ $\square$

③ $[iD_j, X_k] f = (iD_j X_k - X_k iD_j) f$

$= iD_j(x_k f) - X_k i(-i\partial_j f)$

$= i(-i\partial_j)(x_k f) - x_k(\partial_j f)$

$= (\partial_j x_k) f + x_k(\partial_j f) - x_k \partial_j f$

$= (\partial_j x_k) f$

$= \delta_{jk} f \Leftrightarrow \underline{[iD_j, X_k] = \delta_{jk}}$ $\square$