Nguyen Song Ha Jan 22, 2020

Special Math Lectures Report

1, Prove that if $(A, D(A))$ is densely defined in $\mathcal{H}$, then:

a) $\text{Ker}(A^*) = \text{Ran}(A)^\perp$

b) If $(A, D(A)) \subset (B, D(B))$ then $(B^*, D(B^*)) \subset (A^*, D(A^*))$.

Proof:

a) Let $f \in \text{Ker}(A^*)$ then $A^* f = 0$

$\Rightarrow \quad \langle A^* f, g\rangle = 0 \quad \forall g \in D(A)$

$\Rightarrow \quad \langle f, Ag\rangle = \langle A^* f, g\rangle = 0 \quad \forall g \in D(A)$

So $f \in \text{Ran}(A)^\perp$ (by definition). Hence $\text{Ker}(A^*) \subset \text{Ran}(A)^\perp$ (1)

Now let $v \in \text{Ran}(A)^\perp$, it means $\langle v, Ag\rangle = 0 \quad \forall g \in D(A)$

By definition: $D(A^*) = \{ w \in \mathcal{H} \mid \exists! w^* := A^* w$ such that $\langle w^*, g\rangle = \langle w, Ag\rangle \quad \forall g \in D(A)\}$

It is easy to see that if one takes $v^* = 0$ then one has:

$\langle v^*, g\rangle = 0 = \langle v, Ag\rangle \quad \forall g \in D(A)$

Hence $v \in D(A^*)$ and $A^* v = 0$, i.e. $v \in \text{Ker}(A^*)$.

Thus $\text{Ran}(A)^\perp \subset \text{Ker}(A^*)$ (2)

From (1), (2) one has $\text{Ker}(A^*) = \text{Ran}(A)^\perp$ $\square$

b) $(A, D(A)) \subset (B, D(B))$ means that $\begin{cases} D(A) \subset D(B) \\ Af = Bf \quad \forall f \in D(A) \end{cases}$

Now let $g \in D(B^*)$, $\forall f \in D(A)$ one has:

$\langle g, Af\rangle = \langle g, Bf\rangle = \langle B^* g, f\rangle$

$\Rightarrow \quad \langle B^* g, f\rangle = \langle g, Af\rangle \quad \forall f \in D(A)$

Hence by definition, $g \in D(A^*)$ and $A^* g = B^* g$

Thus $(B^*, D(B^*)) \subset (A^*, D(A^*))$ $\square$

2, In $\mathcal{S}(\mathbb{R}^n)$, let $[X_j f](x) = x_j f(x)$ and $[D_j f](x) = -i\partial_j f(x)$

Prove that: $[X_j, X_k] = 0$; $[D_j, D_k] = 0$; $[iD_j, X_k] = \delta_{jk}$.

Proof: Consider $\{[X_j, X_k] f\}(x) = [X_j X_k f - X_k X_j f](x)$

$= x_j x_k f(x) - x_k x_j f(x)$

$= x_k x_j f(x) - x_k x_j f(x) = 0$ $\square$

(since x_j, x_k are just numbers)

$\{[D_j, D_k] f\}(x) = (-\partial_j \partial_k + \partial_k \partial_j) f(x)$

$= -\partial_k \partial_j f(x) + \partial_k \partial_j f(x) = 0$ $\square$

(one can exchange the order of partial derivatives because $f \in \mathcal{S}(\mathbb{R}^n)$ which means f is smooth)

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\begin{aligned}
\{[iD_j, X_k] f\}(x) &= i(-i\partial_j)[x_k f(x)] - x_k\, i(-i\partial_j) f(x) \\
&= \partial_j [x_k f(x)] - x_k \partial_j f(x) \\
&= (\partial_j x_k) f(x) + x_k \partial_j f(x) - x_k \partial_j f(x) \\
&= \delta_{jk} f(x)
\end{aligned}
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Hence $[iD_j, X_k] = \delta_{jk}$ $\square$