Nagoya University, G30 program Fall 2019

Calculus I Instructor : Serge Richard

Midterm

Exercise 1 *Compute the following limits (and justify your computations) :*

1. $\lim_{h\to 0} \frac{\sqrt{9+h}-3}{h}$
2. $\lim_{x\to -2} \frac{2-|x|}{2+x}$
3. $\lim_{x\to 0} \frac{\sin(4x)}{\sin(6x)}$
4. $\lim_{h\to 0} \frac{\frac{1}{(x+h)^2}-\frac{1}{x^2}}{h}$, *for* $x \neq 0$

Exercise 2 *Compute the derivative of the following functions, and simplify the results (if possible) :*

1. $f : \mathbb{R}_+ \to \mathbb{R},\ f(x) = \frac{\sqrt{x}-1}{\sqrt{x}+1}$,
2. $g : \mathbb{R} \to \mathbb{R},\ g(x) = x^3 \sin^2(x)\cos(x)$,
3. $h : \mathbb{R} \to \mathbb{R},\ h(x) = \mathrm{e}^{m(x)} + \big(n(x)\big)^2$, *for any* $m, n \in C^1(\mathbb{R})$.

Exercise 3 *State the* Mean value theorem *as precisely as possible, and draw a picture corresponding to the statement. In the proof of this theorem, a certain function plays a key role, what is this function ?*

Exercise 4 *Consider the function* $\tanh : \mathbb{R} \to \mathbb{R}$ *defined by* $\tanh(x) := \frac{\sinh(x)}{\cosh(x)} = \frac{\mathrm{e}^x - \mathrm{e}^{-x}}{\mathrm{e}^x + \mathrm{e}^{-x}}$.

1. *Determine the range of this function,*
2. *Show that this function is invertible,*
3. *Compute the derivative of its inverse,*
4. *Show that* $\tanh(y)^{-1} = \frac{1}{2}\ln\left(\frac{1+y}{1-y}\right)$ *for any* y *in the range of* f.

Exercise 5 *Consider the curve given by the equation* $(x^2)^{1/3} + (y^2)^{1/3} = 4$.

1. *Find the equation of the tangent to this curve at* $(-3\sqrt{3}, 1)$,
2. *Consider* (x, y) *with* $x, y > 0$ *and* $F(x, y) = 0$. *Find the slope of the tangent to the curve when* (x, y) *approach* $(8, 0)$,
3. *Sketch this curve.*

Exercise 6 *For any* $x \in \mathbb{R}$ *with* $x \neq -1$ *we consider the sequence* $(a_n)_{n\in\mathbb{N}}$ *given by*

$$a_n := \frac{x^n - 1}{1 + x^n}.$$

For which x *does the limit* $a_\infty := \lim_{n\to\infty} a_n$ *exist ? Give the value of this limit whenever it exists. Represent your findings on a graph (the horizontal axis corresponds to the* x*-variable, the vertical axis to the values of* a_∞*).*

Midterm

Total : 22 pts

Exercise 1 4 pts

1) $\dfrac{\sqrt{9+h}-3}{h} = \dfrac{9+h-9}{h(\sqrt{9+h}+3)} = \dfrac{1}{\sqrt{9+h}+3} \xrightarrow{h \to 0} \underline{\dfrac{1}{6}}$.

2) $\lim\limits_{x \to -2} \dfrac{2-|x|}{2+x} = \lim\limits_{x \to -2} \dfrac{2+x}{2+x} = \underline{1}$.

3) $\lim\limits_{x \to 0} \dfrac{\sin(4x)}{\sin(6x)} = \lim\limits_{x \to 0} \left(\dfrac{\sin(4x)}{4x}\right) \cdot 4 \cdot \left(\dfrac{6x}{\sin(6x)}\right) \cdot \dfrac{1}{6} = \underline{\dfrac{2}{3}}$.

4) This is the definition of $\left(\dfrac{1}{x^2}\right)'$, and the answer is $\underline{-2\dfrac{1}{x^3}}$.

Exercise 2 3 pts

1) $f'(x) = \dfrac{\frac{1}{2}x^{-1/2}(x^{1/2}+1) - \frac{1}{2}x^{-1/2}(x^{1/2}-1)}{(x^{1/2}+1)^2} = \dfrac{x^{-1/2}}{(x^{1/2}+1)^2}$.

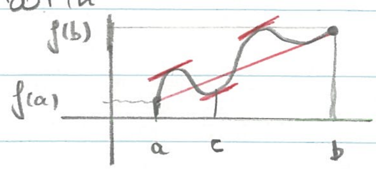
2) $g'(x) = 3x^2 \sin^2(x)\cos(x) + 2x^3 \sin(x)\cos^2(x)$
$\quad - x^3 \sin^3(x)$.

3) $h'(x) = e^{m(x)} m'(x) + 2n(x)\, n'(x)$.

Exercise 3 4 pts

For any $f : [a,b] \to \mathbb{R}$, continuous, and differentiable in (a,b), there exists $c \in (a,b)$ with

$f'(c) = \dfrac{f(b)-f(a)}{b-a}$. 2



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Key function : $h : [a,b] \to \mathbb{R}$

$h(x) = f(x) - \left(\dfrac{f(b)-f(a)}{b-a}(x-a) + f(a)\right)$. 1

Exercise 4   4 pts

1) 1 Since $e^{x} - e^{-x} < e^{x} + e^{-x}$ and
$-(e^{x}+e^{-x}) < e^{x} - e^{-x}$ $\forall x \in \mathbb{R}$,
one infers that $|e^{x} - e^{-x}| \leq e^{x} + e^{-x}$ $\forall x$
$\Leftrightarrow$ $\frac{e^{x}-e^{-x}}{e^{x}+e^{-x}} \in [-1, 1]$.

Since $\lim_{x \to \infty} \frac{e^{x} - e^{-x}}{e^{x} + e^{-x}} = 1$ and

$\lim_{x \to -\infty} \frac{e^{x} - e^{-x}}{e^{x} + e^{-x}} = -1$, and since

the function is continuous, one gets $\mathrm{Ran}(f) = (-1, 1)$

⚠ the values ± 1 are never reached.

2) 1 One has $\tanh'(x) = \left(\frac{\sinh(x)}{\cosh(x)}\right)' = \frac{\cosh^2(x) - \sinh^2(x)}{\cosh^2(x)}$

$= 1 - \tanh^2(x) > 0$ $\forall x \in \mathbb{R}$,

or $= \frac{1}{\cosh^2(x)} > 0$ $\forall x \in \mathbb{R}$.

$\Rightarrow$ the function tanh is strictly increasing,

and thus invertible. Its inverse is denoted by arctanh.

3) 1 $\operatorname{arctanh}'(y) = \frac{1}{\tanh'(\operatorname{arctanh}(y))} = \frac{1}{1 - \tanh(\operatorname{arctanh}(y))^2} = \frac{1}{1-y^2}$

$\forall y \in (-1, 1)$.

4)1 Several possible methods. One is:
set $g(y) := \frac{1}{2} \ln \frac{1+y}{1-y}$ and $f(y) := \operatorname{arctanh}(y)$,
and observe that $g(0) = \frac{1}{2} \ln(1) = 0 = f(0)$.
In addition,

$$g'(y) = \frac{1}{2}\left(\frac{1+y}{1-y}\right)^{-1} \frac{1(1-y)-(-1)(1+y)}{(1-y)^2} = \frac{1}{2}\,\frac{1-y}{1+y}\,\frac{2}{(1-y)^2} = \frac{1}{1-y^2}.$$

Thus f and g are equal at one point ($y=0$) and have same derivative everywhere
$\Rightarrow$ $f = g$ (you can consider $f-g$ with $(f-g)' = 0$ and $(f-g)(0) = 0$ $\Rightarrow$ $f-g = 0$).

Exercise 5 4 pts

1) Set $F(x,y) := (x^2)^{1/3} + (y^2)^{1/3} - 4$ and suppose that
1 locally, the curve defined by $F(x,y) = 0$ can be obtained by $y = f(x)$.
Then

$$F(x, f(x))' = \frac{1}{3}(x^2)^{-2/3}\, 2x + \frac{1}{3}\left(f(x)^2\right)^{-2/3} 2 f(x) f'(x)$$
$$= 0 \quad \text{because} \quad F(x, f(x)) = 0$$

$$\Rightarrow f'(x) = -\frac{(x^2)^{-2/3}\, x}{\left(f(x)^2\right)^{-2/3} f(x)}.$$

Since $F(-3\sqrt{3}, 1) = 3 + 1 - 4 = 0$, $(-3\sqrt{3}, 1)$ belongs to the curve, and one has

$$f'(-3\sqrt{3}) = -\frac{\frac{1}{9}(-3\sqrt{3})}{1} = \frac{\sqrt{3}}{3}.$$

Tangent of the curve at $(-3\sqrt{3}, 1)$:

$$y = \frac{\sqrt{3}}{3}(x + 3\sqrt{3}) + 1.$$

Other methods are possible

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2)$_1$ If $(x,y) \in \mathbb{R}_+ \times \mathbb{R}_+$ and satisfy $F(x,y) = 0$,

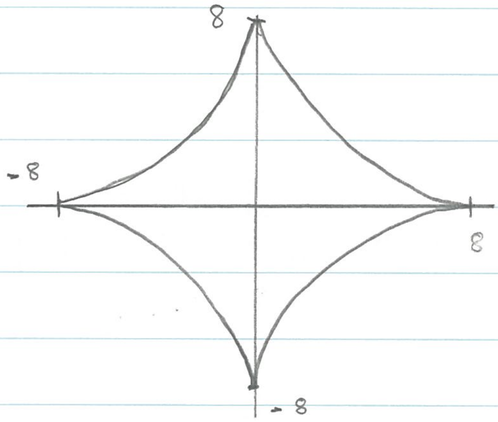
then $y = f(x) > 0$ and $\left(f(x)^2\right)^{-2/3} f(x) = f(x)^{-1/3}$.

$\Rightarrow f'(x) = -\dfrac{x^{-1/3}}{f(x)^{-1/3}} = \dfrac{f(x)^{1/3}}{x^{1/3}}$.

When $(x,y) \to (8,0)$, one has $f^{1/3}(x) \longrightarrow 0$,

$x^{1/3} \longrightarrow 2$, and thus $f'(x) \to 0$. 1

3)$_1$



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Exercise 6 3 pts

1). If $|x| < 1$, then $x^n \xrightarrow{n \to \infty} 0$, and $a_n \xrightarrow{n \to \infty} -1$.

2) If $x = 1$, $a_n = 0$

3) If $|x| > 1$, then $\dfrac{x^n - 1}{1 + x^n} = \dfrac{1 - 1/x^n}{1 + 1/x^n} \xrightarrow{n \to \infty} 1$.