Exercise on Functional Analysis Zhang Liyang

In Weyl Calculus, for $h : \mathbb{R}^n \times \mathbb{R}^n \mapsto \mathbb{C}$ and $u \in L^2(\mathbb{R}^n)$,

$$[h(X,D)u](x) := \frac{1}{(2\pi)^n} \int_{\mathbb{R}^n} \int_{\mathbb{R}^n} e^{i(x-y)\cdot\xi} h\left(\frac{x+y}{2}, \xi\right) u(y)\, dy\, d\xi$$

(1) If $h(X,D)$ depends only on X, i.e.

$\exists \varphi : \mathbb{R}^n \mapsto \mathbb{C}\ \forall p, q \in \mathbb{R}^n : h(p,q) = \varphi(p)$

Then

$$[h(X,D)u](x) = \frac{1}{(2\pi)^n} \int_{\mathbb{R}^n} \int_{\mathbb{R}^n} e^{i(x-y)\cdot\xi} \underbrace{\varphi\left(\frac{x+y}{2}\right) u(y)}_{\overset{!!}{f_x(y)}}\, dy\, d\xi$$

$$= \frac{1}{(2\pi)^n} \int_{\mathbb{R}^n} d\xi\, e^{ix\cdot\xi} \int_{\mathbb{R}^n} dy\, e^{-iy\cdot\xi} f_x(y)$$

$$= \frac{1}{(2\pi)^{n/2}} \int_{\mathbb{R}^n} d\xi\, e^{ix\cdot\xi}\, \hat{f}_x(\xi)$$

$$= [\mathcal{F}^{-1} \hat{f}_x](x) = f_x(x) = \varphi\left(\frac{x+x}{2}\right) u(x)$$

$$= \varphi(x)\, u(x) = [\varphi(X)u](x) \qquad \forall u \in L^2(\mathbb{R}^n), x \in \mathbb{R}^n$$

which means

$h(X,D) = \varphi(X)$

(2) If $h(X,D)$ depends only on D, i.e.

$\exists \varphi : \mathbb{R}^n \mapsto \mathbb{C}\ \forall p, q \in \mathbb{R}^n : h(p,q) = \varphi(q)$

Then

$$[h(X,D)u](x) = \frac{1}{(2\pi)^n} \int_{\mathbb{R}^n} \int_{\mathbb{R}^n} e^{i(x-y)\cdot\xi} \varphi(\xi) u(y)\, dy\, d\xi$$

$$= \frac{1}{(2\pi)^n} \int_{\mathbb{R}^n} d\xi\, e^{ix\xi} \varphi(\xi) \int_{\mathbb{R}^n} dy\, e^{-iy\xi} u(y)$$

$$= \frac{1}{(2\pi)^{n/2}} \int_{\mathbb{R}^n} d\xi\, e^{ix\xi} \varphi(\xi)\, \hat{u}(\xi)$$

$$= [\mathcal{F}^{-1}(\varphi \hat{u})](x) = [(\mathcal{F}^{-1} \varphi \mathcal{F}) u](x)$$

$$= [\varphi(D)u](x) \qquad \forall u \in L^2(\mathbb{R}^n), x \in \mathbb{R}^n$$

which means

$h(X,D) = \varphi(D)$