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Schwarz Inequality in Hilbert space: $|\langle f,g\rangle| \leq \|f\|\cdot\|g\|$.

Proof: case 1: $f=g$. $\Rightarrow \langle f,f\rangle = \|f\|^2$. (by the defination).

case 2: $f\neq g$, and $\|g\|\neq 0$. for any $\alpha\in\mathbb{C}$:

$$0\leq \|f+\alpha g\|^2 = \langle f+\alpha g, f+\alpha g\rangle = \|f\|^2 + \alpha\langle f,g\rangle + \bar{\alpha}\langle g,f\rangle + |\alpha|^2\|g\|^2.$$

Taking $\alpha = -\langle f,g\rangle/\|g\|^2$:

$$\Rightarrow 0\leq \|f\|^2 - \frac{1}{\|g\|^2}|\langle f,g\rangle|^2 \Rightarrow |\langle f,g\rangle| \leq \|f\|\cdot\|g\|.$$

Triangle Iequality: (1). $\|f+g\|\leq\|f\|+\|g\|$.

Proof: $\|f+g\|^2 = \langle f+g, f+g\rangle = \|f\|^2 + \langle f,g\rangle + \langle g,f\rangle + \|g\|^2 \leq \|f\|^2 + |\langle f,g\rangle| + |\langle g,f\rangle| + \|g\|^2$

$\leq \|f\|^2 + 2\|f\|\cdot\|g\| + \|g\|^2 = (\|f\|+\|g\|)^2$.

$\Rightarrow \|f+g\|\leq\|f\|+\|g\|$.

(2). $\|f+g\|^2 \leq 2\|f\|^2 + 2\|g\|^2$.

Proof: $0\leq\|f-g\|^2 = \langle f-g, f-g\rangle = \|f\|^2 - \langle f,g\rangle - \langle g,f\rangle + \|g\|^2$. $\Rightarrow \langle f,g\rangle + \langle g,f\rangle \leq \|f\|^2+\|g\|^2$.

Thus $\|f+g\|^2 = \|f\|^2 + \langle f,g\rangle + \langle g,f\rangle + \|g\|^2 \leq 2\|f\|^2 + 2\|g\|^2$.

(3). $|\|f\|-\|g\|| \leq \|f-g\|$.

Proof: $\|f\|-\|g\| = \|f-g+g\| - \|g\| \leq \|f-g\| + \|g\| - \|g\| = \|f-g\|$. by supposing that $\|f\|>\|g\|$.

$\Rightarrow |\|f\|-\|g\|| \leq \|f-g\|$.

Lemma: for $\{f_n\}_{n\in\mathbb{N}}\subset\mathcal{H}$, one has: $\underset{n\to\infty}{s\text{-}\lim} f_n = f_\infty \Leftrightarrow \underset{n\to\infty}{w\text{-}\lim} f_n = f_\infty$ and $\lim_{n\to\infty}\|f_n\| = \|f_\infty\|$.

Proof: 「$\Rightarrow$」: Assume that $\underset{n\to\infty}{s\text{-}\lim} f_n = f$.

$|\langle f_n,g\rangle - \langle f,g\rangle| = |\langle f_n-f, g\rangle| \leq \|f_n-f\|\cdot\|g\| \to 0$ as $n\to\infty \Rightarrow \underset{n\to\infty}{w\text{-}\lim} f_n = f$.

On the other hand, $|\|f_n\|-\|f\|| \leq \|f_n-f\| \to 0$ as $n\to\infty \Rightarrow \lim_{n\to\infty}\|f_n\| = \|f\|$.

「$\Leftarrow$」: $\|f_n-f\|^2 = \langle f_n-f, f_n-f\rangle = \|f_n\|^2 - \langle f_n,f\rangle - \langle f,f_n\rangle + \|f\|^2$.

Assume that $\underset{n\to\infty}{w\text{-}\lim} f_n = f_\infty$ and $\lim_{n\to\infty}\|f_n\| = \|f_\infty\|$, the expression on the right hand side converges to 0. $\Rightarrow \underset{n\to\infty}{s\text{-}\lim} f_n = f$.