Standard inequalities: $|\langle f,g\rangle| \leqslant \|f\|\,\|g\|$ (1) (Schwartz inequality)

$(f\in \mathcal{H};\ g\in\mathcal{H})$ $\|f+g\| \leqslant \|f\|+\|g\|$ (2) (Triangular inequality)

$\|f+g\|^2 \leqslant 2\|f\|^2+2\|g\|^2$ (3)

$\big|\|f\|-\|g\|\big| \leqslant \|f-g\|$ (4)

Proof:

$\forall \alpha\in\mathbb{C}:\ 0\leqslant \|f+\alpha g\|^2 = \langle f+\alpha g, f+\alpha g\rangle = \langle f+\alpha g, f\rangle + \alpha\langle f+\alpha g, g\rangle = \|f\|^2+\bar{\alpha}\langle g,f\rangle+\alpha\langle f,g\rangle + |\alpha|^2\|g\|^2$ (*)

+ If $g=\mathbf{0}$ then it is obvious that $|\langle f,g\rangle| = 0 = \|f\|\,\|g\|$
+ If $g\neq 0$ then by taking $\alpha = -\dfrac{\overline{\langle f,g\rangle}}{\|g\|^2}$, one gets $0\leqslant \|f+\alpha g\|^2 = \|f\|^2 - \dfrac{\langle f,g\rangle\langle g,f\rangle}{\|g\|^2} - \dfrac{\overline{\langle f,g\rangle}\langle f,g\rangle}{\|g\|^2} + \dfrac{|\overline{\langle f,g\rangle}|^2}{\|g\|^4}\|g\|^2$

  Moreover, $\langle f,g\rangle\langle g,f\rangle = \overline{\langle f,g\rangle}\langle f,g\rangle = |\langle f,g\rangle|^2 = |\overline{\langle f,g\rangle}|^2$

  $\Rightarrow 0\leqslant \|f\|^2 - \dfrac{|\langle f,g\rangle|^2}{\|g\|^2} \Rightarrow |\langle f,g\rangle|^2 \leqslant \|f\|^2\|g\|^2 \Rightarrow |\langle f,g\rangle| \leqslant \|f\|\,\|g\|$

$\Rightarrow \forall f,g\in\mathcal{H}:\ |\langle f,g\rangle| \leqslant \|f\|\,\|g\|$ (1)

Taking $\alpha = 1$ in (*) yields
$$\begin{aligned}\|f+g\|^2 &= \|f\|^2 + \langle g,f\rangle + \langle f,g\rangle + \|g\|^2 \quad (**)\\ &\leqslant \|f\|^2 + |\langle g,f\rangle| + |\langle f,g\rangle| + \|g\|^2\\ &\leqslant \|f\|^2 + 2\|f\|\,\|g\| + \|g\|^2 \quad (\text{using } (1))\\ &= (\|f\|+\|g\|)^2\end{aligned}$$
$\Rightarrow \|f+g\| \leqslant \|f\|+\|g\|$ (2)

Taking $\alpha=-1$ in (*) yeilds: $0\leqslant \|f-g\|^2 = \|f\|^2 - \langle g,f\rangle - \langle f,g\rangle + \|g\|^2 \Rightarrow \langle g,f\rangle + \langle f,g\rangle \leqslant \|f\|^2+\|g\|^2$

$\Rightarrow \|f+g\|^2 \underset{\substack{\uparrow\\ \text{(due to (**))}}}{=} \|f\|^2 + (\langle g,f\rangle + \langle f,g\rangle) + \|g\|^2 \leqslant \|f\|^2 + (\|f\|^2+\|g\|^2) + \|g\|^2$

$\Rightarrow \|f+g\|^2 \leqslant 2\|f\|^2 + 2\|g\|^2$ (3)

If $\|f\| \geqslant \|g\|$ then $\big|\|f\|-\|g\|\big| = \|f\|-\|g\| = \|(f-g)+g\| - \|g\| \underset{\substack{\uparrow\\ \text{due to (2)}}}{\leqslant} (\|f-g\|+\|g\|) - \|g\| = \|f-g\|$

If $\|f\| < \|g\|$ then $\big|\|f\|-\|g\|\big| = \|g\|-\|f\| \leqslant \|g-f\| = \|f-g\|$

$\Rightarrow \forall f,g\in\mathcal{H}:\ \big|\|f\|-\|g\|\big| \leqslant \|f-g\|$ (4)