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Exercise: Prove that $m^*(I) = vol_n(I)$, where I is a closed box in $\mathbb{R}^n$

Before proving the given exercise, we are going to do some preliminary proofs in part a, b, c, d and e.

a) Define an open-closed interval in $\mathbb{R}^n$:

$$H := (a_1, b_1] \times (a_2, b_2] \times \ldots \times (a_n, b_n]$$

$\forall m \in \{1,\ldots,n\}$ set $a_m = c_m^{(0)} < c_m^{(1)} < \ldots < c_m^{(r_m)} = b_m$

Set $H_{i_1,\ldots,i_n} = (c_1^{(i_1-1)}, c_1^{(i_1)}] \times \ldots \times (c_n^{(i_n-1)}, c_n^{(i_n)}]$: open-closed interval

$(1 \leqslant i_m \leqslant r_m \ \ \forall m \in \{1,\ldots,n\})$

Let $\overline{H} = [a_1, b_1] \times [a_2, b_2] \times \ldots \times [a_n, b_n]$ (the closed box corresponding to H)

and $\overline{H}_{i_1,\ldots,i_n} = [c_1^{(i_1-1)}, c_1^{(i_1)}] \times \ldots \times [c_n^{(i_n-1)}, c_n^{(i_n)}]$ (the closed box corresponding to $H_{i_1,\ldots,i_n}$)

Prove that $vol_n(\overline{H}) = \sum\limits_{1\leqslant i_1 \leqslant r_1,\ldots, 1\leqslant i_n \leqslant r_n} vol_n(\overline{H}_{i_1,\ldots,i_n})$

Proof:

$$vol_n(\overline{H}) = (b_1 - a_1)\ldots(b_n - a_n) = \left[\sum_{i_1=1}^{r_1} (c_1^{(i_1)} - c_1^{(i_1-1)})\right] \ldots \left[\sum_{i_n=1}^{r_n} (c_n^{(i_n)} - c_n^{(i_n-1)})\right]$$

$$= \sum_{1\leqslant i_1\leqslant r_1,\ldots, 1\leqslant i_n \leqslant r_n} (c_1^{(i_1)} - c_1^{(i_1-1)})\ldots(c_n^{(i_n)} - c_n^{(i_n-1)}) = \sum_{1\leqslant i_1 \leqslant r_1, \ldots 1\leqslant i_n \leqslant r_n} vol_n(\overline{H}_{i_1,\ldots,i_n}) \quad \text{(Q.E.D)}$$

b) Let $H = (a_1, b_1] \times (a_2, b_2] \times \ldots \times (a_n, b_n]$ and $\overline{H} = [a_1, b_1] \times [a_2, b_2] \times \ldots \times [a_n, b_n]$

Set $\begin{cases} H_j = (a_1^{(j)}, b_1^{(j)}] \times \ldots \times (a_n^{(j)}, b_n^{(j)}] \\ \overline{H}_j = [a_1^{(j)}, b_1^{(j)}] \times \ldots \times [a_n^{(j)}, b_n^{(j)}] \end{cases}$ for $j \in \{1,\ldots,t\}$ and $H_1, \ldots, H_t$ are pairwise disjoint.

Prove that if $H = \bigcup\limits_{j=1}^{t} H_j$ then $vol_n(\overline{H}) = \sum\limits_{j=1}^{t} vol_n(\overline{H}_j)$

Proof:

For any $m \in \{1,\ldots,n\}$ set $C_m = \{c_m^{(0)}, c_m^{(1)}, \ldots, c_m^{(r_m)}\} = \{a_m^{(1)}, \ldots, a_m^{(t)}, b_m^{(1)}, \ldots, b_m^{(t)}\}$ such that $a_m = c_m^{(0)} < c_m^{(1)} < \ldots < c_m^{(r_m)} = b_m$ ($a_m = c_m^{(0)}$ and $b_m = c_m^{(r_m)}$ are because $H = \bigcup\limits_{j=1}^{t} H_j$)

One can observe that there is no repetition in C_m

Let $H_{i_1,\ldots,i_n} = (c_1^{(i_1-1)}, c_1^{(i_1)}] \times \ldots (c_n^{(i_n-1)}, c_n^{(i_n)}]$ with $1 \leqslant i_1 \leqslant r_1, \ldots, 1 \leqslant i_n \leqslant r_n$

and $\overline{H}_{i_1,\ldots,i_n} = [c_1^{(i_1-1)}, c_1^{(i_1)}] \times \ldots \times [c_n^{(i_n-1)}, c_n^{(i_n)}]$

We have some observations:

+ Every $(a_m^{(j)}, b_m^{(j)}]$ is a union of some successive intervals among $(c_m^{(0)}, c_m^{(1)}], \ldots, (c_m^{(r_m-1)}, c_m^{(r_m)}]$
+ As we have subdivided each dimension of H (and also H_j, $j \in \{1,\ldots,t\}$) to get $H_{i_1,\ldots,i_n}$ ($1 \leqslant i_m \leqslant r_m$ $\forall m \in \{1,\ldots,n\}$) we have each H_j a union of some $H_{i_1,\ldots,i_n}$ and H a union of all $H_{i_1,\ldots,i_n}$.

  Moreover, $H_1, \ldots, H_t$ are pairwise disjoint $\Rightarrow$ Each $H_{i_1,\ldots,i_n}$ is included in exactly one from $H_1, \ldots H_t$

According to part a, one gets $vol_n(\overline{H}) = \sum\limits_{1\leqslant i_1 \leqslant r_1,\ldots, 1\leqslant i_n \leqslant r_n} vol_n(\overline{H}_{i_1,\ldots,i_n})$ and $\sum\limits_{j=1}^{t} vol_n(\overline{H}_j) = \sum\limits_{j=1}^{t} \sum\limits_{H_{i_1,\ldots,i_n} \subseteq H_j} vol_n(\overline{H}_{i_1,\ldots,i_n})$

In addition, grouping $H_{i_1,\ldots,i_n}$ included in the same H_j yields $\sum\limits_{1\leqslant i_1\leqslant r_1,\ldots, 1\leqslant i_n \leqslant r_n} vol_n(\overline{H}_{i_1,\ldots,i_n}) = \sum\limits_{j=1}^{t} \sum\limits_{H_{i_1,\ldots,i_n} \subseteq H_j} vol_n(\overline{H}_{i_1,\ldots,i_n})$

$\Rightarrow vol_n(\overline{H}) = \sum\limits_{j=1}^{t} vol_n(\overline{H}_j)$

(c) Let $H = (a_1, b_1] \times \ldots \times (a_n, b_n]$ and $\overline{H} = [a_1, b_1] \times \ldots \times [a_n, b_n]$

Set $\begin{cases} H_j = (a_1^{(j)}, b_1^{(j)}] \times \ldots \times (a_n^{(j)}, b_n^{(j)}] \\ \overline{H_j} = [a_1^{(j)}, b_1^{(j)}] \times \ldots \times [a_n^{(j)}, b_n^{(j)}] \end{cases}$ for $j \in \{1, \ldots, t\}$ and $H_1, \ldots, H_t$ are pairwise disjoint

Prove that if $\bigcup_{j=1}^{t} H_j \subseteq H$ then $\sum_{j=1}^{t} \mathrm{vol}_n(\overline{H_j}) \leqslant \mathrm{vol}_n(\overline{H})$

Proof:

We use the following proposition:

For H and H_j ($j \in \{1, \ldots, t\}$) are open-closed intervals with $H_1, \ldots, H_t$ pairwise disjoint, there exists some open-closed intervals $H_1', \ldots, H_t'$ such that $H \setminus \left(\bigcup_{j=1}^{t} H_j\right) = \bigcup_{j=1}^{t} H_j'$

$\Rightarrow H = \left(\bigcup_{j=1}^{t} H_j\right) \cup \left(\bigcup_{j=1}^{t} H_j'\right)$

Define $\overline{H_j'}$ to be the closed box corresponding to H_j' for $j \in \{1, \ldots, t\}$

According to part b, one has $\mathrm{vol}_n(\overline{H}) = \sum_{j=1}^{t} \mathrm{vol}_n(\overline{H_j}) + \sum_{j=1}^{t} \mathrm{vol}_n(\overline{H_j'}) \geqslant \sum_{j=1}^{t} \mathrm{vol}_n(\overline{H_j})$ (Q.E.D)

(d) Let $H = (a_1, b_1] \times \ldots \times (a_n, b_n]$ and $\overline{H} = [a_1, b_1] \times \ldots \times [a_n, b_n]$

Let $H_j = (a_1^{(j)}, b_1^{(j)}] \times \ldots \times (a_n^{(j)}, b_n^{(j)}]$ and $\overline{H_j} = [a_1^{(j)}, b_1^{(j)}] \times \ldots \times [a_n^{(j)}, b_n^{(j)}]$ for $j \in \{1, \ldots, t\}$

Prove that if $H \subseteq \bigcup_{j=1}^{t} H_j$ then $\mathrm{vol}_n(\overline{H}) \leqslant \sum_{j=1}^{t} \mathrm{vol}_n(\overline{H_j})$

Proof:

Let $H = \bigcup_{j=1}^{t} H_j'$ with $H_j' = H_j \cap H$, which are open-closed intervals.

Then H can be written as a union of t pairwise disjoint sets as follows:

$H = H_1' \cup (H_2' \setminus H_1') \cup \ldots \cup (H_t' \setminus (H_1' \cup H_2' \cup \ldots \cup H_{t-1}'))$

Each of these t pairwise disjoint set can be written as a finite union of pairwise disjoint open-closed intervals.

$H_1' = H_1'$

$H_j' \setminus (H_1' \cup \ldots \cup H_{j-1}') = H_1^{(j)} \cup \ldots \cup H_{s_j}^{(j)} \quad (2 \leqslant j \leqslant t)$

$\Rightarrow H = H_1' \cup (H_1^{(2)} \cup \ldots \cup H_{s_2}^{(2)}) \cup \ldots \cup (H_1^{(t)} \cup \ldots \cup H_{s_t}^{(t)})$

Using corresponding closed boxes, one can obtain: $\mathrm{vol}_n(\overline{H}) \underset{\text{part b}}{=} \mathrm{vol}_n(\overline{H_1'}) + \sum_{j=2}^{t} \sum_{s=1}^{s_j} \mathrm{vol}_n(\overline{H_s^{(j)}})$

$$\overset{\text{part c}}{\leqslant} \mathrm{vol}_n(\overline{H_1'}) + \sum_{j=2}^{t} \mathrm{vol}_n(\overline{H_j'}) = \sum_{j=1}^{t} \mathrm{vol}_n(\overline{H_j'})$$

$$\leqslant \sum_{j=1}^{t} \mathrm{vol}_n(\overline{H_j})$$

$\Rightarrow \mathrm{vol}_n(\overline{H}) \leqslant \sum_{j=1}^{t} \mathrm{vol}_n(\overline{H_j})$ (Q.E.D)

e) Let I be a closed box: $I = [a_1, b_1] \times \ldots \times [a_n, b_n]$

Let $H = (a_1, b_1] \times \ldots \times (a_n, b_n]$ and $\overline{H} := I$ (I is the closed box corresponding to H)

Let H_j be open-closed intervals ($j \in \{1, \ldots, t\}$) such that $I \subset \bigcup_{j=1}^{t} H_j$

and T_j be closed boxes corresponding to H_j

Prove that $\mathrm{vol}_n(I) \leqslant \sum_{j=1}^{t} \mathrm{vol}_n(T_j)$

Proof: $\left.\begin{array}{l} H \subset I \\ I \subset \bigcup_{j=1}^{t} H_j \end{array}\right\} \Rightarrow H \subset \bigcup_{j=1}^{t} H_j \Rightarrow \mathrm{vol}_n(I) \leqslant \sum_{j=1}^{t} \mathrm{vol}_n(T_j)$ (due to part d) (Q.E.D)

Prove that $m^*(I) = vol_n(I)$, where I is a closed box in $\mathbb{R}^n$

Proof

Let $I = [a_1, b_1] \times \dots \times [a_n, b_n]$

$\forall \varepsilon > 0$, $J_\varepsilon := [a_1 - \varepsilon, b_1 + \varepsilon] \times \dots \times [a_n - \varepsilon, b_n + \varepsilon]$ is a covering of I.

J_ε is a closed box with $vol_n(J_\varepsilon) = \prod_{j=1}^{n} (b_j - a_j + 2\varepsilon)$

$\Rightarrow m^*(I) \leqslant vol_n(J_\varepsilon) = \prod_{j=1}^{n} (b_j - a_j + 2\varepsilon)$

ε is arbitrary $\Rightarrow$ $m^*(I) \leqslant \prod_{j=1}^{n} (b_j - a_j) = vol_n(I) \Rightarrow m^*(I) \leqslant vol_n(I)$ (1)

Let $\{H_j\}_{j \in \mathbb{N}}$ be a set of open-closed intervals such that $I \subset \bigcup_j H_j$

Let T_j be the closed box corresponding to H_j $\Rightarrow I \subset \bigcup_j H_j \subset \bigcup_j T_j$

Because I is compact, I can be covered by a finite number of open intervals and thus I can be covered by a finite number of closed-open intervals, which means $\exists t \in \mathbb{N}$ such that: $I \subset \bigcup_{j=1}^{t} H_j$

Due to part e, one can get $vol_n(I) \leqslant \sum_{j=1}^{t} vol_n(T_j)$ $(*)$

Moreover, $T = \{T_j\}_{j=1}^{t}$ is a covering of I. Then by taking the infimum of the right hand side in $(*)$, one has $vol_n(I) \leqslant \inf_{T} \sum_{j=1}^{t} vol_n(T_j) = m^*(I) \Rightarrow vol_n(I) \leqslant m^*(I)$ (2)

Using (1) and (2) gives $vol_n(I) = m^*(I)$ (Q.E.D)