Exercise:

Prove that if $f(x)=\begin{cases} 1 & \text{if } x\in[0,1]\cap\mathbb{Q} \\ 0 & \text{if } x\in[0,1]\setminus\mathbb{Q}\end{cases}$ then f is Lebesgue integrable and $\int_0^1 f(x)dx=0$

Proof: 2 ways to prove that f is Lebesgue integrable are shown as follows:

Method 1: Using the theorem that for $f\in\mathcal{L}^\infty([a,b])$, f is Lebesgue integrable iff f is Lebesgue measurable

- If $s\geqslant 1$ then $\{x\in[0;1]\mid f(x)>s\}$ is $\emptyset$

  $\forall\varepsilon>0\ \exists W$ open with $\emptyset\subset W\subset\mathbb{R}$ such that $m^*(W\setminus\emptyset)=m^*(W)\leqslant\varepsilon\Rightarrow\emptyset$ is Lebesgue measurable

- If $0\leqslant s<1$ then $\{x\in[0;1]\mid f(x)>s\}$ is $[0;1]\cap\mathbb{Q}=V$, which is a countably infinite set.

  $V=\bigcup_j E_j$ with $E_j=\{e_j\}$; $e_j\in[0;1]\cap\mathbb{Q}$

  Let $\varepsilon>0$, arbitrary and $W_j=\left(e_j-\frac{\varepsilon}{2};e_j+\frac{\varepsilon}{2}\right)\supset E_j$

  $m^*(W_j\setminus E_j)=m^*\left((e_j-\frac{\varepsilon}{2},e_j)\cup(e_j,e_j+\frac{\varepsilon}{2})\right)\leqslant m^*(e_j-\frac{\varepsilon}{2},e_j)+m^*(e_j,e_j+\frac{\varepsilon}{2})=\frac{\varepsilon}{2}+\frac{\varepsilon}{2}=\varepsilon$

  $\Rightarrow\forall\varepsilon>0\ \exists W_j$ open with $E_j\subset W_j\subset\mathbb{R}$ such that $m^*(W_j\setminus E_j)\leqslant\varepsilon$

  $\Rightarrow E_j$ is Lebesgue measurable

  $\Rightarrow V=\bigcup_j E_j$ is Lebesgue measurable

- If $s<0$ then $\{x\in[0;1]\mid f(x)>s\}$ is $[0;1]$, which is a Lebesgue measurable set.

$\Rightarrow\forall s\in\mathbb{R}$, the set $\{x\in[0;1]\mid f(x)>s\}$ is Lebesgue measurable

$\Rightarrow f$ is Lebesgue measurable (by def.) $\Leftrightarrow f$ is Lebesgue integrable (Q.E.D)

Method 2: Using definition

Consider a measurable partition P of $[0;1]$ consisting of a finite collection $\{E_j\}_{j\in\mathbb{N}}\subset[0;1]$

$$\sup_P L(f,P)=\sup_P\sum_{j=1}^N\left(\inf_{E_j}f(x)\right)m(E_j)$$

If $E_j\subset[0,1]\cap\mathbb{Q}$ then $\inf_{E_j}(f(x))=1$ and $m(E_j)=\sum_k\underbrace{m(\{e_k\})}_{=0}=0$ with $e_k\in E_j$

Otherwise, E_j contains some irrational numbers in $[0;1]\Rightarrow\inf_{E_j}(f(x))=0$

$\Rightarrow L(f,P)=0\ \forall P\Rightarrow\sup_P L(f,P)=0$

$$U(f,P)=\sum_{j=1}^N\left(\sup_{E_j}f(x)\right)m(E_j)$$

$\sup_{E_j}f(x)\in\{0;1\}$ and $m(E_j)\geqslant 0$ result in $U(f,P)\geqslant 0$

Moreover, $\exists P$ such that $U(f,P)=0$.

For example, $U=E_1\cup E_2$ with $\begin{cases}E_1=[0,1]\cap\mathbb{Q}\Rightarrow m(E_1)=0\\ E_2=[0,1]\setminus\mathbb{Q}\Rightarrow\sup_{E_j}f(x)=0\end{cases}$

$\Rightarrow\inf_P U(f,P)=0$

Therefore, $\sup_P L(f,P)=\inf_P U(f,P)$, which means that f is Lebesgue integrable (Q.E.D)

Based on method 2, one has $\int_0^1 f(x)dx=\sup_P L(f,P)=\inf_P L(f,P)=0$ (Q.E.D)