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Date No.

1) Show that principal value $\left(\text{p.v.}\frac{1}{x}\right) \in \mathcal{D}'(\mathbb{R})$

Proof:

By definition, for $f \in \mathcal{D}(\mathbb{R})$

$$\left(\text{p.v.}\frac{1}{x}\right)(f) = \lim_{\varepsilon \to 0^+}\left[\int_{-\infty}^{-\varepsilon} \frac{1}{x} f(x)dx + \int_{\varepsilon}^{+\infty} \frac{1}{x} f(x)dx\right]$$

Let $t = -x \Rightarrow \left(\text{p.v.}\frac{1}{x}\right)(f) = \lim_{\varepsilon \to 0^+}\left[-\int_{\varepsilon}^{+\infty} \frac{1}{t} f(-t)dt + \int_{\varepsilon}^{+\infty} \frac{1}{x} f(x)dx\right] = \lim_{\varepsilon \to 0^+} \int_{\varepsilon}^{+\infty} \frac{f(x) - f(-x)}{x} dx$

Assume that $\text{supp}(f) \subset [-a; a]$ $(a > 0)$

Then $\left(\text{p.v.}\frac{1}{x}\right)(f) = \lim_{\varepsilon \to 0^+} \int_{\varepsilon}^{a} \frac{f(x) - f(-x)}{x} dx$

First, we want to check that $\left(\text{p.v}\frac{1}{x}\right)$ is well-defined.

$$\lim_{x \to 0^+} \frac{f(x) - f(-x)}{x} \overset{\uparrow}{=} \lim_{x \to 0^+} \frac{f'(x) + f'(-x)}{1} = 2f'(0)$$

L'Hospital's rule

$f \in \mathcal{D}(\mathbb{R}) \Rightarrow f \in C^\infty(\mathbb{R}) \Rightarrow f'(x)$ is continuous on $\mathbb{R} \Rightarrow \lim_{x \to 0^+} \frac{f(x) - f(-x)}{x} \left(= 2f'(0)\right)$ exists.

$\Rightarrow \frac{f(x) - f(-x)}{x}$ is continuous at $x = 0$ (*) $\Rightarrow \frac{f(x) - f(-x)}{x}$ is continuous in $\mathbb{R}$

$\Rightarrow \lim_{\varepsilon \to 0^+} \int_{\varepsilon}^{a} \frac{f(x) - f(-x)}{x} dx$ exists $\Rightarrow \left(\text{p.v.}\frac{1}{x}\right)(f)$ is well-defined.

Moreover, $\left(\text{p.v.}\frac{1}{x}\right)(f) = \lim_{\varepsilon \to 0^+} \int_{\varepsilon}^{a} \frac{f(x) - f(-x)}{x}dx = \lim_{\varepsilon \to 0^+} \int_{\varepsilon}^{a} \frac{2x}{x} \frac{f(x) - f(-x)}{2x} dx = \lim_{\varepsilon \to 0^+} \int_{\varepsilon}^{a} 2\frac{f(x) - f(-x)}{x - (-x)}dx$

By mean value theorem, one has for any $x \in [0; \infty)$ $\exists c \in (-x; x)$ such that $f'(c) = \frac{f(x) - f(-x)}{x - (-x)}$

$\sup_{c \in \mathbb{R}} |f'(c)| = \sup_{x \in \mathbb{R}} |f'(x)| = \|f'\|_\infty$

$\Rightarrow \left(\text{p.v.}\frac{1}{x}\right)(f) \leqslant 2 \lim_{\varepsilon \to 0^+} \int_{\varepsilon}^{a} \sup_{c \in \mathbb{R}} |f'(c)| dx = 2a\|f'\|_\infty \leqslant 2a \sum_{|\alpha| \leqslant 1} \|\partial^\alpha f\|_\infty$

$\Rightarrow \forall y \in \mathbb{R}$ and $R > 0$: $\exists c > 0$ and $m \in \mathbb{N}$ such that $\left|\left(\text{p.v.}\frac{1}{x}\right)(f)\right| \leqslant c \sum_{|\alpha| \leqslant m} \|\partial^\alpha f\|_\infty$

for any $f \in \mathcal{D}(\mathbb{R})$ with $\text{supp}(f) \subset (y - R; y + R)$

(1)

Let $g \in \mathcal{D}(\mathbb{R})$ and $\lambda \in \mathbb{R}$, then:

$$\left(\text{p.v.}\frac{1}{x}\right)(f + \lambda g) = \lim_{\varepsilon \to 0^+}\left[\int_{-\infty}^{-\varepsilon} \frac{1}{x} (f + \lambda g)(x)dx + \int_{\varepsilon}^{+\infty} \frac{1}{x} (f + \lambda g)(x)dx\right]$$

$$= \lim_{\varepsilon \to 0^+}\left\{\int_{-\infty}^{-\varepsilon} \frac{1}{x} [f(x) + \lambda g(x)]dx + \int_{\varepsilon}^{+\infty} \frac{1}{x} [f(x) + \lambda g(x)]dx\right\}$$

$$= \left[\lim_{\varepsilon \to 0^+}\left(\int_{-\infty}^{-\varepsilon} \frac{1}{x} f(x)dx + \int_{\varepsilon}^{+\infty} \frac{1}{x} f(x)dx\right)\right] + \lambda\left[\lim_{\varepsilon \to 0^+}\left(\int_{-\infty}^{-\varepsilon} \frac{1}{x} g(x)dx + \int_{\varepsilon}^{+\infty} \frac{1}{x} g(x)dx\right)\right]$$

$$= \left(\text{p.v.}\frac{1}{x}\right)(f) + \lambda\left(\text{p.v.}\frac{1}{x}\right)(g)$$

$\Rightarrow \left(\text{p.v.}\frac{1}{x}\right)$ is linear (2)

According to (1) and (2), one can deduce that $\left(\text{p.v.}\frac{1}{x}\right) \in \mathcal{D}'(\mathbb{R})$

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2) Show that $\ln(|x|) \in L^1_{loc}(\mathbb{R})$

Firstly, it is obvious that $\ln(|x|)$ is locally integrable at any $x \in \mathbb{R} \setminus \{0\}$

Let $0 < b < \infty$

$$\lim_{\varepsilon \to 0^+} \int_\varepsilon^b \ln(x)\,dx = \lim_{\varepsilon \to 0^+} \left(\ln(x)\,x \Big|_\varepsilon^b - \int_\varepsilon^b \ln'(x)\,x\,dx \right) = \left(\lim_{\varepsilon \to 0^+} x\ln(x) \Big|_\varepsilon^b \right) - \lim_{\varepsilon \to 0^+} \int_\varepsilon^b \frac{1}{x}\,x\,dx$$

$$= b\ln(b) - \lim_{\varepsilon \to 0^+} \varepsilon\ln(\varepsilon) - b$$

Moreover, $\displaystyle \lim_{\varepsilon \to 0^+} \varepsilon\ln(\varepsilon) = \lim_{\varepsilon \to 0^+} \frac{\ln(\varepsilon)}{1/\varepsilon} \underset{\uparrow}{=} \lim_{\varepsilon \to 0^+} \frac{\ln'(\varepsilon)}{(1/\varepsilon)'} = \lim_{\varepsilon \to 0^+} \frac{1/\varepsilon}{-1/\varepsilon^2} = \lim_{\varepsilon \to 0^+} (-\varepsilon) = 0$

(L'Hospital's rule)

$\Rightarrow \displaystyle \lim_{\varepsilon \to 0^+} \int_\varepsilon^b \ln(x)\,dx = b\ln(b) - b$, which is finite

Likewise, we can also prove that $\displaystyle \lim_{\substack{\varepsilon \to 0^- \\ (-\infty < a < 0)}} \int_a^\varepsilon \ln(|x|)\,dx$ is finite

$\Rightarrow \ln(|x|)$ is locally integrable at $x = 0$

Therefore, $\ln(|x|) \in L^1_{loc}(\mathbb{R})$