Nagoya University, G30 program Fall 2019

Calculus I Instructor : Serge Richard

Final examination

Exercise 1 *Compute the following limits:*

$$\lim_{x\to 1}\frac{\ln(x)}{x-1}, \qquad \lim_{x\nearrow\pi}\frac{\sin(x)}{1-\cos(x)}, \qquad \lim_{x\to 0}\frac{\tan(x)-x}{x^3}, \qquad \lim_{x\searrow 0}\frac{e^{-1/x^2}}{x^{2020}}.$$

Exercise 2 *Compute the derivative of the following functions (and simplify the results, if possible):*

$$a)\ x\mapsto \frac{1}{\sqrt[3]{x^2+x+1}}, \qquad b)\ x\mapsto\Big(\frac{x-2}{2x+1}\Big)^2, \qquad c)\ x\mapsto (2x+1)^5(x^3-x+1)^4.$$

Exercise 3 *Compute the following integrals:*

$$a)\ \int x^2\big(\ln(x)\big)^2\,\mathrm{d}x, \qquad b)\ \int\frac{\sin(\sqrt{x})}{\sqrt{x}}\,\mathrm{d}x, \qquad c)\ \int_{-1}^{1}\frac{\sin(x)}{1+x^2}\,\mathrm{d}x.$$

Exercise 4 *Compute the value* $e^{0.1}$ *correct to 3 decimal places (= X.XXX). Indication: use Taylor expansion.*

Exercise 5 *Let* $f:[0,1]\to\mathbb{R}_+$ *be continuous and consider the volume of revolution generated by the rotation of* $\big\{\big(x,f(x)\big)\mid x\in[0,1]\big\}$ *around the* x*-axis. Show (by using Riemann sums) that the volume* V *of this solid is given by the expression*

$$V=\pi\int_0^1 f(x)^2\,\mathrm{d}x.$$

Exercise 6 (Alternating series) *1) Recall the criterion of convergence for alternating series.*

2) Let us now prove this criterion. Write the alternating series as $b_1-c_1+b_2-c_2+b_3-c_3+\dots$ *with* $b_j\geq 0$ *and* $c_j\geq 0$ *for any* $j\in\mathbb{N}$*. Set also*

$$s_n := b_1-c_1+b_2-c_2+b_3-c_3+\dots+b_n,$$
$$t_n := b_1-c_1+b_2-c_2+b_3-c_3+\dots+b_n-c_n.$$

Show that $\{s_n\}_{n\in N}$ *is a decreasing sequence, that* $\{t_n\}_{n\in N}$ *is an increasing sequence, and find the relation between* s_n *and* t_n*. Conclude about the convergence of the alternating series.*

Final exam

Total : 26pts

Exercise 1 6pts

1 1) $\lim\limits_{x \to 1} \dfrac{\ln(x)}{x-1} \overset{\text{Hospital}}{=} \lim\limits_{x \to 1} \dfrac{\frac{1}{x}}{1} = \underline{1}.$

1 2) $\lim\limits_{x \nearrow \pi} \dfrac{\sin(x)}{1-\cos(x)} = \dfrac{0}{2} = \underline{0}.$

2 3) Recall $\sin(x) = x - \frac{1}{3!}x^3 + O(x^5)$

$\cos(x) = 1 - \frac{1}{2}x^2 + O(x^4)$

This represents any function satisfying: $\lim\limits_{x \to 0} \dfrac{O(x^n)}{x^n}$ exists

$\Rightarrow \tan(x) = \dfrac{\sin(x)}{\cos(x)} = \left(x - \frac{1}{3!}x^3 + O(x^5)\right)\left(1 - \frac{1}{2}x^2 + O(x^4)\right)^{-1}$

$= \left(x - \frac{1}{3!}x^3 + O(x^5)\right)\left(1 + \frac{1}{2}x^2 + O(x^4)\right)$

$= x - \frac{1}{3!}x^3 + \frac{1}{2}x^3 + O(x^4)$

$= x + \frac{1}{3}x^3 + O(x^4)$

The use of L'Hospital is also possible.

Thus $\lim\limits_{x \to 0} \dfrac{\tan(x) - x}{x^3} = \underline{\dfrac{1}{3}}.$

2 4) $\lim\limits_{x \searrow 0} \dfrac{e^{-1/x^2}}{x^{2020}} \underset{\text{set } y = 1/x}{=} \lim\limits_{y \to \infty} \dfrac{e^{-y^2}}{y^{-2020}} = \lim\limits_{y \to \infty} \dfrac{y^{2020}}{e^{y^2}} = \underline{0},$

because the exponential goes faster to ∞ than any polynomial.

Exercise 2   6 pts

2 a) $\left((x^2+x+1)^{-\frac{1}{3}}\right)' = -\frac{1}{3}(x^2+x+1)^{-4/3}(2x+1)$

$$= -\frac{2x+1}{3(x^2+x+1)^{4/3}}.$$

2 b) $\left(\left(\frac{x-2}{2x+1}\right)^2\right)' = 2\,\frac{x-2}{2x+1}\,\frac{2x+1-2(x-2)}{(2x+1)^2}$

$$= 10\,\frac{(x-2)}{(2x+1)^3}.$$

2 c) $\left((2x+1)^5(x^3-x+1)^4\right)' =$

$$= 5(2x+1)^4 2(x^3-x+1)^4 + (2x+1)^5 4(x^3-x+1)^3(3x^2-1)$$

$$= (2x+1)^4(x^3-x+1)^3\left[10(x^3-x+1) + 4(2x+1)(3x^2-1)\right]$$

$$= (2x+1)^4(x^3-x+1)^3(34x^3+12x^2-18x+6)$$

$$= 2(2x+1)^4(x^3-x+1)^3(17x^3+6x^2-9x+3).$$

both answer accepted.

Exercise 3 4 pts

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2 a) $\int \overset{f'}{x^2} (\ln(x))^2 \, dx = \frac{1}{3} x^3 (\ln(x))^2 - \frac{2}{3} \int x^2 \ln(x) \, dx$

$= \frac{1}{3} x^3 (\ln(x))^2 - \frac{2}{9} x^3 \ln(x) + \frac{2}{9} \int x^2 \, dx$

$= \frac{1}{3} x^3 (\ln(x))^2 - \frac{2}{9} x^3 \ln(x) + \frac{2}{27} x^3 + \text{cst}.$

1 b) $\int \frac{\sin(\sqrt{x})}{\sqrt{x}} \, dx$ set $\sqrt{x} = y$ $\frac{1}{2} \frac{1}{\sqrt{x}} \, dx = dy$

$= 2 \int \sin(y) \, dy = -2 \cos(y) = -2 \cos(\sqrt{x}).$

1 c) $\int_{-1}^{1} \frac{\sin(x)}{1+x^2} \, dx = 0$ because it is an odd function on a symmetric domain.

Exercise 4 3 pts

Set $f(x) = e^x$. Then $f(\frac{1}{10}) = 1 + \frac{1}{1!} \frac{1}{10} + \frac{1}{2!} \left(\frac{1}{10}\right)^2$

$+ \frac{1}{3!} \left(\frac{1}{10}\right)^3 + R_4(0, \frac{1}{10})$

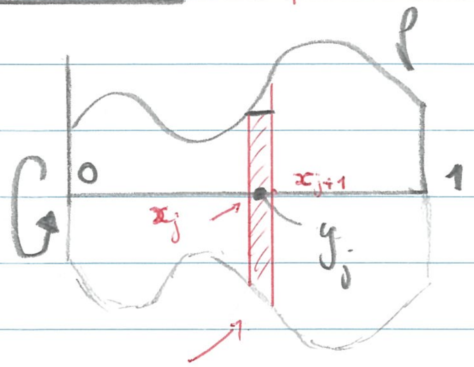
$= 1 + 0{,}1 + 0{,}005 + \underbrace{\frac{1}{6} \cdot 0{,}001}_{\simeq 0{,}1\ldots} + R_4(0, \frac{1}{10})$ smaller than 0,001

$= \underline{1{,}105}$

and $R_4(0,1) \leq \frac{1}{4!} \left(\frac{1}{10}\right)^4 \underbrace{e^{\frac{1}{10}}}_{< 2}$ which is thus negligible.

(1 point for R_4)

Exercise 5 2 pts



Interval $[0,1]$ divided in n sub-intervals of length $\frac{1}{n}$

volume $\simeq \frac{1}{n} \times \pi \, f(y_j)^2$ volume of a thin disk

Total volume $\cong \sum_{j=0}^{n-1} \frac{1}{n} \pi \, f(y_j)^2$

$= \pi \sum_{j=0}^{n-1} f(y_j)^2 \, (x_{j+1} - x_j)$ which is the definition of the middle Riemann sum for the function f^2.

$\xrightarrow{n \to \infty} \pi \int_0^1 f(x)^2 \, dx$

by definition of the limit of Riemann sum, when it exists. Since f and f^2 are continuous, the limit exists.

Exercise 6   5 pts

2 1) A series $\sum_{j=1}^{\infty} a_j$ is alternating if

1) $a_j a_{j+1} \leq 0$
2) $|a_j| \geq |a_{j+1}|$
3) $\lim_{j \to \infty} |a_j| = 0$ (or $\lim_{j \to \infty} a_j = 0$)

Thm: any alternating series is convergent.

3 2) Proof. Set the series $b_1 - c_1 + b_2 - c_2 + \dots$,

possible because of 1)

$S_n := b_1 - c_1 + b_2 - c_2 + \dots + b_n$,

$t_n := S_n - c_n$.

Observe that $S_{n+1} = S_n \underbrace{- c_n + b_{n+1}}_{\leq 0,\ \text{by } 2)} \leq S_n$.

Also $t_{n+1} = t_n + \underbrace{b_{n+1} - c_{n+1}}_{\geq 0,\ \text{by } 2)} \geq t_n$

$\Rightarrow \{S_n\}$ is decreasing, $\{t_n\}$ is increasing.

Also, since $t_n = S_n - c_n \Rightarrow t_n \leq S_n$. We get

$$\begin{array}{ccccccccc} S_1 & \geq & S_2 & \geq & S_3 & \geq & S_4 & \dots \\ \vee\!| & & \vee\!| & & \vee\!| & & \vee\!| & \\ t_1 & \leq & t_2 & \leq & t_3 & \leq & t_4 & \dots \end{array}$$

$\Rightarrow \{t_n\}$ is an increasing and upper bounded sequence

by S_1

$\Rightarrow$ it converges,

and $\{S_n\}$ is a decreasing and lower bounded sequence => it converges.
by t1

Since $S_n - t_n = c_n \xrightarrow{n \to \infty} 0$
by 3)

it follows that $\lim_{n \to \infty} S_n = \lim_{n \to \infty} t_n$,

which means that the 2 sequences converge to the same limit. $\square$