Nagoya University, G30 program Fall 2019

Mathematics Tutorial Ia (calculus) Instructor : Serge Richard

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Homework 7

Exercise 1 *Compute the following limits:*

(i) $\lim_{x\to 0} \frac{x-\sin(x)}{x^3}$,

(ii) $\lim_{x\to 0} \frac{x^2}{1+x-\mathrm{e}^x}$.

Exercise 2 *Find the critical points for the differentiable functions* $f : \mathbb{R} \to \mathbb{R}$ *defined for* $x \in \mathbb{R}$ *by*

$$a)\ -x^2+2x+2, \qquad b)\ x^3-3, \qquad c)\ \cos(x), \qquad d)\ \sin(x)+\cos(x).$$

Exercise 3 *Find the point of the curve of equation* $y^2 = 4x$ *which is the nearest one to the point* $(2,3)$.

Exercise 4 *Prove the following statement:* Let $f : [a,b] \to \mathbb{R}$ be a strictly increasing and continuous function, and set $\alpha := f(a)$ and $\beta := f(b)$. Then there exists an inverse function $f^{-1} : [\alpha,\beta] \to [a,b]$ such that $f^{-1}\big(f(x)\big) = x$ and $f\big(f^{-1}(y)\big) = y$ for any $x \in [a,b]$ and $y \in [\alpha,\beta]$.

Recall that we have shown the existence of a unique differentiable function $f : \mathbb{R} \to \mathbb{R}$ satisfying $f'(x) = f(x)$ for any $x \in \mathbb{R}$ and $f(0) = 1$. In addition, it has been shown that this function satisfies the following properties:

(i) $f(x)f(-x) = 1$ for any $x \in \mathbb{R}$,

(ii) $f(x) > 0$ for any $x \in \mathbb{R}$,

(iii) f is strictly increasing,

(iv) $f(x+y) = f(x)f(y)$ for any $x, y \in \mathbb{R}$.

From now on, the notation e^x for $f(x)$ will be used. The previous relations read $\mathrm{e}^x\mathrm{e}^{-x} = 1$, $\mathrm{e}^x > 0$, $x \mapsto \mathrm{e}^x$ is strictly increasing, and $\mathrm{e}^{x+y} = \mathrm{e}^x\mathrm{e}^y$. It follows then from the previous exercise that this function is invertible. This inverse is denoted by ln. More precisely, $\ln : \mathbb{R}_+ \to \mathbb{R}$ satisfies $\mathrm{e}^{\ln(y)} = y$ and $\ln(\mathrm{e}^x) = x$ for any $x \in \mathbb{R}$ and $y \in \mathbb{R}_+$.

Exercise 5 *Prove the following properties of the function* ln *:*

(i) $\ln(y)' = \frac{1}{y}$ *for any* $y \in \mathbb{R}_+$,

(ii) $\ln(yz) = \ln(y) + \ln(z)$ *for any* $y, z \in \mathbb{R}_+$,

(iii) $\ln(y^x) = x\ln(y)$ *for any* $y \in \mathbb{R}_+$ *and* $x \in \mathbb{Q}$.