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§ The manifold structure on TM

I will make $TM = \bigsqcup_{p\in M} T_pM$ into a smooth manifold.

The first step is to give it a topology.

Rmk. An element of TM can be thought of as a pair (p, X_p), where $p \in M$ and $X_p \in T_pM$.

(1) The topology of TM

Let $(U, \phi) = (U, x^1, \dots, x^n)$ be a coordinate chart of M.

Recall: At $p \in U$, $\left\{ \frac{\partial}{\partial x^1}\Big|_p, \dots, \frac{\partial}{\partial x^n}\Big|_p \right\}$ is a basis for T_pM

A tangent vector $X_p \in T_pM$ is uniquely determined by

$$X_p = \sum_i^n a^i \frac{\partial}{\partial x^i}\Big|_p$$

where $a^i = a^i(X_p) \in \mathbb{R}$ depends on X_p.

Let $TU = \bigsqcup_{p\in U} T_pM$

We define $\tilde{\phi} = (\phi, \phi_*)$

$$\tilde{\phi} = (\phi, \phi_*) : TU \longrightarrow \phi(U) \times \mathbb{R}^n \subset \mathbb{R}^{2n}$$

$$(p, X_p) \longmapsto (x^1(p), \dots, x^n(p), a^1(X_p), \dots, a^n(X_p))$$

Prop. $\tilde{\Phi}$ is bijective, with inverse

$$(\phi(p), a^1, \dots, a^n) \mapsto (p, \sum a^i \frac{\partial}{\partial x^i}\Big|_p)$$

We define an open set in TU by using $\tilde{\Phi}$:

a set A in TU is open iff $\tilde{\Phi}(A)$ is open in $\phi(U) \times \mathbb{R}^n$

Then we give a basis for some topology on TM

Let $\mathcal{B}$ be the collection of all open subsets of $T(U_\alpha)$ as U_α runs over all coordinate open sets in M.

Lem. Let U and V be coordinate open sets in M
If A is open in TU and if B is open in TV,
then $A \cap B$ is open in $T(U \cap V)$

proof Since $T(U \cap V)$ is a subspace of TU and TV,

$A \cap T(U \cap V)$ and $B \cap T(U \cap V)$ are open in $T(U \cap V)$

But $A \cap B \subset TU \cap TV = T(U \cap V)$

Hence $A \cap B = (A \cap T(U \cap V)) \cap (B \cap T(U \cap V))$

is open in $T(U \cap V)$. $\square$

It follows from this lemma, that $\mathcal{B}$ satisfies the following conditions:

(i) M is the union of all the sets in $\mathcal{B}$

(ii) Given B_1 and $B_2 \in \mathcal{B}$ and $p \in B_1 \cap B_2$,

there is a set $B \in \mathcal{B}$ such that $p \in B \subset B_1 \cap B_2$

Then we define an open set in M:

$A \subset M$ is open iff there exists $\{B_\lambda\} \subset \mathcal{B}$ s.t. $A = \bigcup_\lambda B_\lambda$

Fact. M has a countable basis consisting of coordinate open sets

Idea of proof of this fact

Let $\{(U_\alpha, \phi_\alpha)\}$ be the maximal atlas on M and

$\mathcal{B} = \{B_i\}$ a countable basis for M.

For each U_α and $p \in U_\alpha$, choose $B_{p,\alpha} \in \mathcal{B}$ such that

$p \in B_{p,\alpha} \subset U_\alpha$

Then $\{B_{p,\alpha}\}$ is what we want. $\square$

Prop. TM is second countable

proof. Let $\{U_i\}_{i=1}^{\infty}$ be a countable basis of M consisting of coordinate open sets.

Since $TU_i \underset{\text{homeo}}{\simeq} \phi_i(U_i) \times \mathbb{R}^n \underset{\text{open}}{\subset} \mathbb{R}^{2n}$, TU_i is second countable.

For each i, choose a countable basis $\{B_{i,j}\}_{j=1}^{\infty}$ for TU_i

Then $\{B_{i,j}\}_{i,j=1}^{\infty}$ is a countable basis for TM. □

Prop. TM is Hausdorff

proof Let (p, X) and (q, Y) be distinct points of TM.

Case 1 : $p \neq q$

Since M is Hausdorff, there are open sets U and V of M $p \in U$, $q \in V$ and $U \cap V = \phi$.

Then TU and TV are disjoint open subsets of TM. containing (p, X) and (q, Y) respectively.

Case 2 : $p = q$

Let U be a coordinate neighborhood of p.

Then (p, X) and (p, Y) are distincts points in $TU \simeq \phi(U) \times \mathbb{R}^n$, which is Hausdorff.

So (p, X) and (q, Y) can be separated in TU $\square$

Because of these properties, TM becomes a topological manifold.

Next I check that TM is smooth manifold.

(III) The manifold structure on TM

Prop. Let $\{(U_\alpha, \phi_\alpha)\}$ be an atlas for M.

Then $\{(TU_\alpha, \tilde{\phi}_\alpha)\}$ is an atlas for TM

It is clear that $TM = \bigcup_\alpha TU_\alpha$

It remains to check that on $TU_\alpha \cap TU_\beta$, $\tilde{\phi}_\alpha$ and $\tilde{\phi}_\beta$ are C^∞ compatible.

Obsv. Let $(U, x^1, \ldots, x^n)$ and $(V, y^1, \ldots, y^n)$ be two charts on M. Then for any $p \in U \cap V$, there are two bases for T_pM : $\{\frac{\partial}{\partial x^j}|_p\}_{j=1}^n$ and $\{\frac{\partial}{\partial y^i}|_p\}_{i=1}^n$

So X_p has two descriptions :

$$X_p = \sum_j a^j \frac{\partial}{\partial x^j}\Big|_p = \sum_i b^i \frac{\partial}{\partial y^i}\Big|_p$$

By applying y^k on both sides gives

$$b^k = \sum_j a^j \frac{\partial y^k}{\partial x^j}$$

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proof of prop.

$$\tilde{\Phi}_\beta \circ \tilde{\Phi}_\alpha^{-1} : \phi_\alpha(U_{\alpha\beta}) \times \mathbb{R}^n \longrightarrow \phi_\beta(U_{\alpha\beta}) \times \mathbb{R}^n$$

is given by

$$(x, a^1, \dots, a^n) \mapsto \left(\phi_\alpha^{-1}(x), \sum_j a^j \frac{\partial}{\partial x^j}\Big|_p\right)$$

$$\mapsto \left(\phi_\beta \circ \phi_\alpha^{-1}(x), b^1, \dots, b^n\right),$$

where $b^i = \sum_j a^j \frac{\partial y^i}{\partial x^j}$

By the definition of an atlas, $\phi_\beta \circ \phi_\alpha^{-1}$ is C^∞;

It means that the y^i-components are C^∞ functions of the x^j-components.

This implies $\frac{\partial y^i}{\partial x^j}$ are C^∞ functions.

Therefore $\tilde{\Phi}_\beta \circ \tilde{\Phi}_\alpha^{-1}$ is C^∞.

Similarly, $\tilde{\Phi}_\alpha \circ \tilde{\Phi}_\beta^{-1}$ is C^∞ $\square$