06170178 Tomoya Tatsuno

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Exercise from Oct. 3 (Special mathematics lecture)

Def. Let $(M, \mathcal{T}), (N, \mathcal{S})$ be 2 topological spaces and let $f: M \to N$

f is continuous if $\forall U \in \mathcal{S}, f^{-1}[U] \in \mathcal{T}$

When $M = N = \mathbb{R}$ and $\mathcal{T} = \mathcal{S} = \{\text{open sets in } \mathbb{R}\}$, check if this def corresponds to the ε-δ def of continuousity.

<answer>

(i) f is continuous in the meaning of this def.

(ii) f is continuous in the meaning of the ε-δ def.

(i) $\Leftrightarrow$ (ii)

First,

(ii)

$\Leftrightarrow \forall \varepsilon > 0, \forall a \in \mathbb{R}, \exists \delta > 0$ s.t. $|x - a| < \delta \Rightarrow |f(x) - f(a)| < \varepsilon$

$\Leftrightarrow \forall \varepsilon > 0, \forall a \in \mathbb{R}, \exists \delta > 0$ s.t. $x \in B(a, \delta) \Rightarrow f(x) \in B(f(a), \varepsilon)$

$\Leftrightarrow \forall \varepsilon > 0, \forall a \in \mathbb{R}, \exists \delta > 0$, s.t. $f[B(a, \delta)] \subset B(f(a), \varepsilon)$

$\Leftrightarrow \forall \varepsilon > 0, \forall a \in \mathbb{R}, \exists \delta > 0$ s.t. $B(a, \delta) \subset f^{-1}[B(f(a), \varepsilon)]$

[(i) $\Rightarrow$ (ii)]

For any $\varepsilon > 0$, and for any $a \in \mathbb{R}$, $B(f(a), \varepsilon)$ is an open set in $\mathbb{R}$, then $B(f(a), \varepsilon) \in \mathcal{S}$.

By (i), $f^{-1}[B(f(a), \varepsilon)] \in \mathcal{T}$.

$a \in f^{-1}[B(f(a), \varepsilon)]$, and $f^{-1}[B(f(a), \varepsilon)]$ is an open set in $\mathbb{R}$,

so there exists $\delta > 0$ such that $B(a, \delta) \subset f^{-1}[B(f(a), \varepsilon)]$

Therefore, for any $\varepsilon > 0$, and for any $a \in \mathbb{R}$, there exists $\delta > 0$ such that $B(a, \delta) \subset f^{-1}[B(f(a), \varepsilon)]$

[(ii) $\Rightarrow$ (i)]

Let U be an arbitrary open set in $\mathcal{S}$.

[I] If $U = \emptyset$,

$f^{-1}[U] = \emptyset \in \mathcal{T}$

[II] If $U \neq \emptyset$,

$f^{-1}[U] \in \mathcal{T}$

$\Leftrightarrow \forall x \in f^{-1}[U]\ \exists \delta > 0$ s.t. $B(x, \delta) \subset f^{-1}[U]$

We will show this.

For any $x \in f^{-1}[U]$, $f(x) \in U$.

U is an open set in $\mathbb{R}$, so there exists $\varepsilon > 0$ such that $B(f(x), \varepsilon) \subset U$

By (ii), for this $\varepsilon > 0$, there exists $\delta > 0$ such that $f[B(x, \delta)] \subset B(f(x), \varepsilon) \subset U$

Then $B(x, \delta) \subset f^{-1}[U]$

Therefore, $\forall x \in f^{-1}[U]\ \exists \delta > 0$ s.t. $B(x, \delta) \subset f^{-1}[U]$

Then $f^{-1}[U] \in \mathcal{T}$

Hence $\forall U \in \mathcal{S}, f^{-1}[U] \in \mathcal{T}$ ■