Proofs of Bianchi Identity and some properties of $\langle R(X,Y),Z,W\rangle$

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Assumptions

1. (M,Φ) is a Riemannian Manifold.

2. ∇ is a torsion free connection for M, $\left(\nabla_X Y-\nabla_Y X-[X,Y]=0 \text{ for any smooth } X,Y \text{ for } M\right)$.

3. ∇ is compatible with Φ. $\left(X(\Phi(Y,Z))=\Phi(\nabla_X Y,Z)+\Phi(Y,\nabla_X Z) \text{ for any smooth } X,Y,Z \text{ for } M\right)$.

Given $R(X,Y,Z,W):=\langle R(X,Y)Z,W\rangle$ and $R(X,Y):=\nabla_X\nabla_Y-\nabla_Y\nabla_X-\nabla_{[X,Y]}$ is antisymmetric.

To prove

1.$R(X,Y)Z+R(Y,Z)X+R(Z,X)Y=0$;

2.$R(X,Y,Z,W)=-R(Y,X,Z,W)$; 3. $R(X,Y,Z,W)=-R(X,Y,W,Z)$;4. $R(X,Y,W,Z)=R(W,Z,X,Y)$

with X, Y, Z, W being arbtary smooth vector fields of M:

1. $\mathbf{R(X,Y)Z+R(Y,Z)X+R(Z,X)Y=0}$ **(Bianchi Identity)**

$R(X,Y)Z+R(Y,Z)X+R(Z,X)Y$

$=\left(\nabla_X\nabla_Y Z-\nabla_Y\nabla_X Z-\nabla_{[X,Y]}Z\right)+\left(\nabla_Y\nabla_Z X-\nabla_Z\nabla_Y X-\nabla_{[Y,Z]}X\right)+\left(\nabla_Z\nabla_X Y-\nabla_X\nabla_Z Y-\nabla_{[Z,X]}Y\right)$

$=\nabla_X\left(\nabla_Y Z-\nabla_Z Y\right)+\nabla_Y\left(\nabla_Z X-\nabla_X Z\right)+\nabla_Z\left(\nabla_X Y-\nabla_Y X\right)-\nabla_{[X,Y]}Z-\nabla_{[Y,Z]}X-\nabla_{[Z,X]}Y)$

$\left(\text{Torsion free} \Rightarrow \nabla_X Y-\nabla_Y X-[X,Y]=0 \Rightarrow \nabla_X Y-\nabla_Y X=[X,Y] \text{ For any smooth } X,Y\right)$

$=\nabla_X[Y,Z]-\nabla_{[Y,Z]}X+\nabla_Y[Z,X]-\nabla_{[Z,X]}Y+\nabla_Z[X,Y]-\nabla_{[X,Y]}Z$

$=\left[X,[Y,Z]\right]+\left[Y,[Z,X,]\right]+\left[Z,[X,Y]\right]$

By Jacobi Identity, $\left[X,[Y,Z]\right]+\left[Y,[Z,X,]\right]+\left[Z,[X,Y]\right]=0 \Rightarrow R(X,Y)Z+R(Y,Z)X+R(Z,X)Y=0$

Define $\langle *,*\rangle=\Phi(*,*)$ for all following $\langle *,*\rangle$ notations.

2. $\mathbf{R(X,Y,Z,W)=-R(Y,X,Z,W)}$:

$R(X,Y,Z,W)=\langle R(X,Y)Z,W\rangle=\langle -R(Y,X)Z,W\rangle=-\langle R(Y,X)Z,W\rangle=-R(Y,X,Z,W)$

3. $\mathbf{R(X,Y,Z,W)=-R(X,Y,W,Z)}$:

$R(X,Y,Z,W)$

$=\left\langle \nabla_X\nabla_Y Z-\nabla_Y\nabla_X Z-\nabla_{[X,Y]}Z,W\right\rangle$

$=\left\langle \nabla_X\nabla_Y Z,W\right\rangle-\left\langle \nabla_Y\nabla_X Z,W\right\rangle-\left\langle \nabla_{[X,Y]}Z,W\right\rangle$

$\left(\text{As } X\langle Y,Z\rangle=\left\langle \nabla_X Y,Z\right\rangle+\left\langle Y,\nabla_X Z\right\rangle, \left\langle \nabla_X Y,Z\right\rangle=X\langle Y,Z\rangle-\left\langle Y,\nabla_X Z\right\rangle \text{ for any smooth } X,Y,Z.\right)$

$=\left(X\left\langle \nabla_Y Z,W\right\rangle-\left\langle \nabla_Y Z,\nabla_X W\right\rangle\right)-\left(Y\left\langle \nabla_X Z,W\right\rangle-\left\langle \nabla_X Z,\nabla_Y W\right\rangle\right)-\left([X,Y]\langle Z,W\rangle-\left\langle Z,\nabla_{[X,Y]}W\right\rangle\right)$

$=\left(X\left(Y\langle Z,W\rangle-\left\langle Z,\nabla_Y W\right\rangle\right)-\left\langle \nabla_Y Z,\nabla_X W\right\rangle\right)-\left(Y\left(X\langle Z,W\rangle-\left\langle Z,\nabla_X W\right\rangle\right)-\left\langle \nabla_X Z,\nabla_Y W\right\rangle\right)-\left([X,Y]\langle Z,W\rangle-\left\langle Z,\nabla_{[X,Y]}W\right\rangle\right)$

$=XY\langle Z,W\rangle-YX\langle Z,W\rangle-X\left\langle Z,\nabla_Y W\right\rangle-\left\langle \nabla_Y Z,\nabla_X W\right\rangle+Y\left\langle Z,\nabla_X W\right\rangle+\left\langle \nabla_X Z,\nabla_Y W\right\rangle-[X,Y]\langle Z,W\rangle+\left\langle Z,\nabla_{[X,Y]}W\right\rangle$

$=[X,Y]\langle Z,W\rangle-\left(\left\langle \nabla_X Z,\nabla_Y W\right\rangle+\left\langle Z,\nabla_X\nabla_Y W\right\rangle\right)-\left\langle \nabla_Y Z,\nabla_X W\right\rangle+\left(\left\langle \nabla_Y Z,\nabla_X W\right\rangle+\left\langle Z,\nabla_Y\nabla_X W\right\rangle\right)+\left\langle \nabla_X Z,\nabla_Y W\right\rangle-[X,Y]\langle Z,W\rangle+\left\langle Z,\nabla_{[X,Y]}W\right\rangle$

$=-\left\langle \nabla_X Z,\nabla_Y W\right\rangle-\left\langle Z,\nabla_X\nabla_Y W\right\rangle-\left\langle \nabla_Y Z,\nabla_X W\right\rangle+\left\langle \nabla_Y Z,\nabla_X W\right\rangle+\left\langle Z,\nabla_Y\nabla_X W\right\rangle+\left\langle \nabla_X Z,\nabla_Y W\right\rangle+\left\langle Z,\nabla_{[X,Y]}W\right\rangle$

$=-\left(\left\langle Z,\nabla_X\nabla_Y W\right\rangle-\left\langle Z,\nabla_Y\nabla_X W\right\rangle-\left\langle Z,\nabla_{[X,Y]}W\right\rangle\right)$

$=-\left\langle Z,\nabla_X\nabla_Y W-\nabla_Y\nabla_X W-\nabla_{[X,Y]}W\right\rangle$

$\left(\text{As } \Phi \text{ is symmetric}\right)$

$=-\left\langle \nabla_X\nabla_Y W-\nabla_Y\nabla_X W-\nabla_{[X,Y]}W,Z\right\rangle$

$=-R(X,Y,W,Z)$

4. $\mathbf{R(X,Y,W,Z)=R(W,Z,X,Y)}$:

$R(X,Y,Z,W)+R(Y,Z,X,W)+R(Z,X,Y,W)$

$=\langle R(X,Y)Z,W\rangle+\langle R(Y,Z)X,W\rangle+\langle R(Z,X)Y,W\rangle$

$=\langle R(X,Y)Z+R(Y,Z)X+R(Z,X)Y,W\rangle$

(Bianchi Identity)

$=\langle 0,W\rangle=0$

$\Rightarrow$

$0+0+0+0=0=$

$[R(X,Y,Z,W)+R(Y,Z,X,W)+R(Z,X,Y,W)]+\left[R(Y,Z,W,X)+R(Z,W,Y,X)+R(W,Y,Z,X)\right]$

$+[R(Z,W,X,Y)+R(W,X,Z,Y)+R(X,Z,W,Y)]+[R(W,X,Y,Z)+R(X,Y,W,Z)+R(Y,W,X,Z)]$

(Result of 3.)

$=R(X,Y,Z,W)+R(Y,Z,X,W)+R(Z,X,Y,W)-R(Y,Z,X,W)-R(Z,W,X,Y)+R(W,Y,Z,X)$

$+R(Z,W,X,Y)+R(W,X,Z,Y)+R(X,Z,W,Y)-R(W,X,Z,Y)-R(X,Y,Z,W)+R(Y,W,X,Z)$

$=R(Z,X,Y,W)+R(W,Y,Z,X)+R(X,Z,W,Y)+R(Y,W,X,Z)$

(Result of 2. and 3.)

$=2R(Z,X,Y,W)-2R(Y,W,Z,X)$

$\Rightarrow$

$2R(Z,X,Y,W)=2R(Y,W,Z,X) \Leftrightarrow R(Z,X,Y,W)=R(Y,W,Z,X) \Leftrightarrow R(X,Y,Z,W)=R(Z,W,X,Y)$ /.