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Theorem: Let $F : M \to N$ be a smooth map between smooth manifolds.
For any $p \in M$ we set: $\forall f \in C^\infty(F(p))$

$$F^*: \quad C^\infty(F(p)) \longrightarrow C^\infty(p) \; ; \; F^*(f) = f \circ F$$

$$F_*: \quad T_p(M) \longrightarrow T_{F(p)}(N) \; ; \; [F_*(X_p)](f) = X_p(F^*(f))$$

Then $F^*$ is a homomorphism of algebra and $F_*$ is a homomorphism of vector space.

Proof:

1) $F^*$ is a homomorphism of algebra iff:

$$\begin{cases} F^*(f + \alpha g) = F^*(f) + \alpha F^*(g) \\ F^*(fg) = F^*(f)\, F^*(g) \end{cases}$$

Firstly, let's be precise about the domain of all the maps above.
$f$ is defined in a neighborhood of $F(p)$, we call it $V_f$
$g$ " " another neighborhood of $F(p)$ " " $V_g$

So $f + \alpha g$ is then defined in the intersection $V_f \cap V_g$

Hence $F^*(f + \alpha g)$ is defined in the preimage region $F^{-1}(V_f \cap V_g)$

For any $q \in F^{-1}(V_f \cap V_g)$ (a neighborhood of $P$) one has:

$$\begin{aligned} F^*(f + \alpha g)(q) &\overset{\text{def}}{=} [(f + \alpha g) \circ F](q) \\ &= (f + \alpha g)(F(q)) \\ &= f(F(q)) + \alpha g(F(q)) \\ &= (f \circ F)(q) + \alpha (g \circ F)(q) \\ &= [F^*(f)](q) + \alpha [F^*(g)](q) \end{aligned}$$

definition of functional addition

Similarly one has:

$$
\begin{aligned}
[F^*(fg)](q) &= (fg \circ F)(q) \\
&= (fg)(F(q)) \\
&= f(F(q))\, g(F(q)) \quad \text{definition of functional multiplication} \\
&= [F^*(f)](q)\,[F^*(g)](q)
\end{aligned}
$$

Therefore $F^*$ is a homomorphism of algebra. □

2) $F_*$ is a homomorphism of vector space iff:

$$F_*(X_p + \alpha Y_p) = F_*(X_p) + \alpha F_*(Y_p)$$

Indeed, by definition one has:

$$
\begin{aligned}
[F_*(X_p + \alpha Y_p)](f) &\overset{def}{=} [X_p + \alpha Y_p](F^*(f)) \\
&= X_p(F^*(f)) + \alpha Y_p(F^*(f)) \quad \text{since } T_p(M) \text{ is a vector space} \\
&= [F_*(X_p)](f) + \alpha [F_*(Y_p)](f) \quad \forall f \in C^\infty(F(p))
\end{aligned}
$$

Therefore $F_*$ is a homomorphism of vector space □