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Let V be a finite dimensional vector space over $\mathbb{R}$ and let V^* be its dual space i.e. the set of all linear maps from V to $\mathbb{R}$. Then $\dim(V^*) = \dim(V)$.

Proof: Let $\dim(V) = n$ and let $\{e_1; \dots; e_n\}$ be a basis of V

Define $e^i \in V^*$: $e^i(e_j) = \delta_{ij}$ $(i, j = 1; \dots; n)$

We will show that $\{e^1; \dots; e^n\}$ is a basis of V^*

Firstly, we need to show that $\{e^1; \dots; e^n\}$ spans V^*

Let $f \in V^*$ and $v \in V$, one has:

$$f(v) = f\left(\sum_j v_j e_j\right)$$

$$= \sum_j v_j f(e_j)$$

Denote $f(e_j) = \lambda_j$

Hence $f(v) = \sum_j v_j \lambda_j e^j(e_j)$

Notice that : $e^j(v) = e^j\left(\sum_k v_k e_k\right)$

$$= \sum_k v_k e^j(e_k)$$

$$= v_j e^j(e_j)$$

So $f(v) = \sum_j \lambda_j e^j(v) = \left[\sum_j \lambda_j e^j\right](v)$

Therefore for any $f \in V^*$, it can be expressed as a linear combination of $\{e^1; \dots; e^n\}$

So $\{e^1; \dots; e^n\}$ spans V^* (1)

Secondly, we need to prove that $e^1; \dots; e^n$ are linearly independent.

Let $\bar{0}$ be the zero-element of V^* and consider :

$c_1 e^1 + \dots + c_n e^n = \bar{0}$

One has: $\bar{0}(e_j) = [\sum_i c_i e^i](e_j)$ (for $j = 1; \dots; n$)

$$= \sum_i c_i e^i(e_j)$$

$$= c_j$$

And by definition $\bar{0}(e_j) = 0$ so $c_j = 0$ ($\forall j = 1; \dots; n$)

So $e^1; \dots; e^n$ are linearly independent (2)

From (1) and (2), $\{e^1; \dots; e^n\}$ is a basis of V^*

And thus $\dim(V^*) = n$ $\square$

Remark: This is true only for finite dimensional vector spaces. For infinite dimensional vector spaces, it's not true in general.