Quang Nhat Nguyen

DIFFERENTIAL GEOMETRY REPORT

1/ About the tangent space

I.1 Proof for the theorem:

Let $F: M \to N$ be a smooth map between smooth manifolds. For $p \in M$, set:

$$F^* : C^\infty(F(p)) \longrightarrow C^\infty(p)$$
$$f \longmapsto f \circ F$$

And $F_* : T_p(M) \longrightarrow T_{F(p)}(N)$

$$[F_*(X_p)](f) = X_p(F^* f) \equiv X_p(f \circ F), \qquad f \in C^\infty(F(p)).$$

Then: F^* is a homomorphism of algebra, and (1)

F_* is a homomorphism of vector space. (2)

(1) F^* is a homeomorphism of algebra with algebraic addition, multiplication of functions in $C^\infty(F(p))$

$$\Leftrightarrow \begin{cases} F^*(f+\alpha g) = F^*(f) + \alpha F^*(g) & \text{(i)} \\ F^*(fg) = F^*(f)\cdot F^*(g) & \text{(ii)} \end{cases} \qquad \left(\begin{array}{c} f, g \in C^\infty(F(p)) \\ \alpha \in \mathbb{R} \end{array}\right)$$

Let us prove that (i) and (ii) are true.

Consider for any $p \in M$:

$$[F^*(f+\alpha g)](p) = (f+\alpha g)(\underbrace{F(p)}_{\in N}) = f(F(p)) + \alpha g(F(p))$$

(from def. of a smooth function from a manifold to $\mathbb{R}$)

$$= [f\circ F](p) + \alpha[g \circ F](p) = [\underbrace{F^*(f)}_{\in C^\infty(p)}](p) + [\underbrace{\alpha F^*(g)}_{\in C^\infty(p)}](p)$$

$$= [F^*(f) + \alpha F^*(g)](p) \quad \forall p \in M \quad \Rightarrow \text{(i) is true}$$

Next, consider:

$$[F^*(fg)](p) = [(fg)\circ F](p) = [fg](\underbrace{F(p)}_{\in N}) = f(F(p))\cdot g(F(p))$$

(from def. as well)

$$= [f\circ F](p)\cdot[g \circ F](p) = [\underbrace{F^*(f)}_{\in C^\infty(p)}](p)\cdot[\underbrace{G^*(g)}_{\in C^\infty(p)}](p)$$

$$= [F^*(f)\cdot F^*(g)](p) \quad \forall p \in M. \Rightarrow \text{(ii) is true.} \qquad \square$$

(2) F_* is a homomorphism of vector space, which is the tangent space in this case with addition and multiplication of tangent vectors

$$\Leftrightarrow F_*(X_p + \alpha Y_p) = F_*(X_p) + \alpha F_*(Y_p), \qquad X_p, Y_p \in T_p(M),\ \alpha \in \mathbb{R}$$

Let us prove the above equality. For $f \in C^\infty(F(p))$:

$$[F_*(X_p + \alpha Y_p)](f) = (X_p + \alpha Y_p)(\underbrace{F^* f}_{\in C^\infty(p)}) \overset{\text{set}}{=} (X_p + \alpha Y_p)(g) \quad (g \in C^\infty(p))$$

$$= X_p(g) + \alpha Y_p(g) \quad \text{(from def. of tangent space)}$$

$$= X_p(F^* f) + \alpha Y_p(F^* f)$$

$$= \underbrace{[F_*(X_p)]}_{\in T_{F(p)}(N)}(f) + \underbrace{[\alpha F_*(Y_p)]}_{\in T_{F(p)}(N)}(f)$$

$$= [F_*(X_p) + \alpha F_*(Y_p)](f) \quad \forall f \in C^\infty(F(p))$$

$$\Rightarrow F_*(X_p + \alpha Y_p) = F_*(X_p) + \alpha F_*(Y_p). \qquad \square$$

Exercise about directional derivative (6.11 (1), Pg. 32, Godinho)

Let $X: M \to TM$ be a differentiable vector field on M, and for a smooth function $f: M \to \mathbb{R}$, consider its directional derivative along X:

$$Xf : M \to \mathbb{R}$$
$$p \mapsto X_p f := X_p(f) \qquad (X_p : C^\infty(M) \to \mathbb{R})$$

Show that:

a) $(Xf)(p) = (df)_p X_p$

One has: $(Xf)(p) = X_p(f) \underset{\substack{\uparrow \\ \text{by def}}}{=} \sum_{j=1}^{n} (X_p)_j \left(\frac{\partial f}{\partial x_j}\right)(p) = X_p \cdot (df)_p = (df)_p X_p.$

b) The vector field X is smooth if and only if Xf is a differentiable function for any smooth function $f: M \to \mathbb{R}$.

"$\Rightarrow$": If X is smooth, then immediately we have Xf is smooth since f is smooth

"$\Leftarrow$": Assume Xf is differentiable.

Let $X_p = \sum_{j=1}^{n} \alpha_j(p) E_{j,p}$. For a coordinate system (U, φ) on M, consider

$$\mathbb{R}^n \supset \varphi(U) \ni x \mapsto (\alpha \circ \varphi^{-1})(x) \in \mathbb{R}^n$$

Let us show that there exists a map of this kind that is smooth.

$Xf(p) = X_p(f) = \sum_{j=1}^{n} (X_p)_j \underbrace{\left(\frac{\partial f}{\partial x_j}\right)}_{\text{local coordinate frame}}(p)$ is differentiable (by assumption)

If we look at the map: $\mathbb{R}^n \supset \varphi(U) \ni x \longmapsto (X_p \circ \varphi^{-1})(x)$, we can see that it is smooth since $X_p : C^\infty(M) \to \mathbb{R}$: smooth. This satisfies the requirement for the smoothness of vector field X.

c) The directional derivative satisfies the following properties: for $f, g \in C^\infty(M)$ and $\alpha \in \mathbb{R}$:

(i) $X(f+g) = Xf + Xg$

(ii) $X(\alpha f) = \alpha(Xf)$

(iii) $X(fg) = f \cdot Xg + g \cdot Xf$

Proof: the general idea is from the result of question (a).

(i) $[X(f+g)](p) = (d(f+g))_p X_p \underset{\uparrow}{=} ((df)_p + (dg)_p) X_p = (df)_p X_p + (dg)_p X_p$

$= (Xf)(p) + (Xg)p \quad \forall p \in M$

property of differential F_*, which has been proved in previous exercise

$\Rightarrow X(f+g) = Xf + Xg$.

(ii) $[X(\alpha f)](p) = (d(\alpha f))_p X_p \underset{\uparrow}{=} \alpha (df)_p X_p = \alpha (Xf)(p). \quad \forall p \in M$

again, the property of F_*

$\Rightarrow X(\alpha f) = \alpha Xf$

(iii) $[X(fg)](p) \stackrel{\text{def}}{=} X_p(fg) \underset{\uparrow}{=} f(p) \cdot X_p(g) + g(p) X_p(f) = [f \cdot Xg + g Xf](p)$

from def. of tangent space

$\Rightarrow X(fg) = fXg + gXf$.

$\square$

Proof of properties of Lie bracket: $[X,Y] = X\circ Y - Y\circ X = \sum_{i=1}^{n} (X\cdot Y^i - Y\cdot X^i)\frac{\partial}{\partial x^i}$

(Godinho, Pg. 28)

Given $X, Y, Z \in \mathfrak{X}(M)$. We have:

(i) Bilinearity: $\forall \alpha, \beta \in \mathbb{R}$:

$$[\alpha X + \beta Y, Z] = \alpha[X,Z] + \beta[Y,Z]$$
$$[X, \alpha Y + \beta Z] = \alpha[X,Y] + \beta[X,Z]$$

(ii) Antisymmetry:

$$[X,Y] = -[Y,X]$$

(iii) Jacobi identity:

$$[[X,Y],Z] + [[Y,Z],X] + [[Z,X],Y] = 0$$

(iv) Leibniz rule: $\forall f, g \in C^\infty$:

$$[fX, Gy] = fg[X,Y] + f(Xg)Y - g(Yf)X$$

Proof:

(i) $[\alpha X + \beta Y, Z] = (\alpha X + \beta Y)\circ Z - Z\circ(\alpha X + \beta Y)$

$= \alpha(X\circ Z) + \beta(Y\circ Z) - \alpha(Z\circ X) - \beta(Z\circ Y)$ (from def. of Lie bracket ↑)

$= \alpha(X\circ Z - Z\circ X) + \beta(Y\circ Z - Z\circ Y)$ (α, β are just real numbers)

$= \alpha[X,Z] + \beta[Y,Z]$.

Similarly: $[X, \alpha Y + \beta Z] = \alpha[X,Y] + \beta[X,Z]$ □

(ii) $[X,Y] = X\circ Y - Y\circ X = -(Y\circ X - X\circ Y) = -[Y,X]$

(iii) $[[X,Y],Z] + [[Y,Z],X] + [[Z,X],Y]$

$= (X\circ Y - Y\circ X)\circ Z - Z\circ(X\circ Y - Y\circ X) + (Y\circ Z - Z\circ Y)\circ X - X\circ(Y\circ Z - Z\circ Y)$

$+ (Z\circ X - X\circ Z)\circ Y - Y\circ(Z\circ X - X\circ Z)$

$= (X\circ Y)\circ Z - (Y\circ X)\circ Z - Z\circ(X\circ Y) + Z\circ(Y\circ X)$

$- X\circ(Y\circ Z) + Y\circ(X\circ Z) + (Z\circ X)\circ Y - (Z\circ Y)\circ X$

$+ (Y\circ Z)\circ X - Y\circ(Z\circ X) + X\circ(Z\circ Y) - (X\circ Z)\circ Y = 0$ □

(iv) $[fX, gY] = (fX)\circ(gY) - (gY)\circ(fX)$

$= f(gX)\circ Y + f(Xg)Y - g(fY)\circ X - g(Yf)X$ (from previous exercise)

$= fg[X,Y] + f(Xg)Y - g(Yf)X$.

Proof of a lemma about alternating tensor :

Let $\phi \in \mathcal{T}^r(V)$ be a tensor : $\phi : V \times V \times \cdots \times V \longrightarrow \mathbb{R}$

Define $\mathcal{A} : \mathcal{T}^r(V) \longrightarrow \mathcal{T}^r(V)$ by :

$$[\mathcal{A}(\phi)](v_1, \ldots, v_r) \equiv [\mathcal{A}\phi](v_1, \ldots, v_r) := \frac{1}{r!} \sum_{\sigma \in S_r} \operatorname{sgn}(\sigma)\, \phi(v_{\sigma(1)}, \ldots, v_{\sigma(r)})$$

in which S_r is the set of all permutation of $\{1, 2, \ldots, r\}$, and: $\sigma : \mathbb{R} \to \mathbb{R}$.

Then : $\mathcal{A}\phi$ is an alternating tensor $\Longleftrightarrow$ $\mathcal{A}\phi \in \Lambda^r(V)$ $\Longleftrightarrow$ $\mathcal{A}\,\mathcal{T}^r(V) = \Lambda^r(V)$

Proof :

Let $\sigma_0 : \mathbb{R} \to \mathbb{R}$ be a permutation that interchanges 2 indicies and keep the others fixed

Then :

$$\operatorname{sgn}(\sigma_0) \cdot \mathcal{A}\phi(v_{\sigma_0(1)}, \ldots, v_{\sigma_0(r)}) = \frac{1}{r!} \sum_{\sigma \in S_R} \operatorname{sgn}(\sigma_0 \circ \sigma)\, \phi(v_{\sigma_0(\sigma(1))}, \ldots, v_{\sigma_0(\sigma(r))})$$

$$= \frac{1}{r!} \sum_{\lambda \in S_R} \operatorname{sgn}(\lambda)\, \phi(v_{\lambda(1)}, \ldots, v_{\lambda(r)})$$

$$= \mathcal{A}\phi(v_1, \ldots, v_r)$$

$$\Rightarrow \mathcal{A}\phi(v_1, \ldots, v_r) = -\mathcal{A}\phi(v_{\sigma_0(1)}, \ldots, v_{\sigma_0(r)})$$ (because by the way we define σ_0, $\operatorname{sgn}(\sigma_0) = -1$)

This indicates that $\mathcal{A}\phi$ changes sign under the permutation of 2 arguments.

Hence, $\mathcal{A}\phi$ is an alternating tensor

$\Longleftrightarrow$ $\mathcal{A}\phi \in \Lambda^r(V)$ $\Longleftrightarrow$ $\mathcal{A}\,\mathcal{T}^r(V) = \Lambda^r(V)$. $\square$