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Special Mathematics lecture : Differential geometry – Lecture 11 Exercises.

Exercise 1:

if $X_p = \sum_{i=1}^{n} b^i(p) E_{i,p}$ and $Y_p = \sum_{j=1}^{n} a^j(p) E_{j,p}$

Prove that $\underbrace{\nabla_X Y = \sum_k \left(X a^k + \sum_{i,j} a^j b^i \Gamma_{i,j}^k \right) E_k}_{(*)}$ defines an affine connection.

Solution :

An affine connection is a Bilinear map on M (a smooth manifold)

$$\nabla : \mathfrak{X}(M) \times \mathfrak{X}(M) \longmapsto \mathfrak{X}(M)$$

$$X \quad Y \longmapsto \nabla(X,Y) \equiv \nabla_X Y \text{ satisfying:}$$

$$\left.\begin{aligned} &1)\ \nabla_{fX} Y = f \nabla_X Y \\ &2)\ \nabla_X (fY) = (Xf) Y + f \nabla_X Y \end{aligned}\right\} \forall f \in C^\infty(M)$$

We shall first show that (*) satisfies 1) :

- $fX = f \sum_{i=1}^{n} b^i(p) E_{i,p} = \sum_{i=1}^{n} (f\, b^i(p)) E_{i,p}$

$$= \sum_{i=1}^{n} c^i(p) E_{i,p}$$

$$c^i(p) = f\, b^i(p)$$

Then, $\nabla_{fX} Y = \sum_k \left(fX a^k + \sum_{i,j} a^j c^i \Gamma_{ij}^k \right) E_k$

$$= \sum_k \left(fXa^k + \sum_{i,j} \underset{\substack{\cap \\ C^\infty(M)}}{a^j}\, \underset{\substack{\cap \\ C^\infty(M)}}{f}\, b^i \Gamma_{ij}^k \right) E_k$$

$$= f \left(\sum_k X a^k + \sum_{i,j} a^j b^i \Gamma_{ij}^k \right) E_k$$

since $a^j, f \in C^\infty(M) \Rightarrow$ they commute.

So we have seen that ⊛ agrees with 1)

Now we shall show that ⊛ satisfies 2)

- $$fY = f\sum_{i=1}^{n} a^i(p)E_{i,p} = \sum_{i=1}^{n}(f a^i(p))E_{i,p}$$
$$= \sum_{i=1}^{n} d^i(p)\,E_{i,p}$$

where, $f a^i(p) = d^i(p)$

Then,

$$\nabla_X(fY) = \sum_k \Big(X(f a^k) + \sum_{i,j} f a^j b^i \Gamma_{ij}^k\Big)E_k$$

$$= \sum_k \Big(\underbrace{(Xf)}_{\mathbb{R}}a^k + f(Xa^k) + f\sum_{i,j} a^j b^i \Gamma_{ij}^k\Big)E_k$$

$$= (Xf)\underbrace{\sum_k a^k E_k}_{Y} + f\sum_k\Big(Xa^k + \sum_{i,j} a^j b^i \Gamma_{ij}^k\Big)E_k$$

$$= (Xf)Y + f\nabla_X Y$$

Thus we have seen that ⊛ agrees with 2)

Since ⊛ agrees with 1) and 2) it defines an affine connection.