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Special Mathematics lecture (Differential geometry) Excercises (Lec - 10)

(if $x, y, z \in \mathfrak{X}(M)$)

Lemma :

(i) $T(x, y)$ is $C^\infty(M)$ - linear in the both arguements.

(ii) $R(x, y)z$ is linear in the 3 arguements.

Where,

$$T(x, y) := \nabla_x y - \nabla_y x - [x, y] \in \mathfrak{X}(M)$$

$$R(x, y) := \nabla_x \nabla_y - \nabla_y \nabla_x - \nabla_{[x,y]} \in \mathrm{End}(\mathfrak{X}(M))$$

Proof:

First we shall prove the following that will be used in the proofs for (i) and (ii).

if $f, g \in C^\infty(M)$ and $x, y \in \mathfrak{X}(M)$

then $[fx, gy] = fg[x, y] + f(x \cdot g)y - g(y \cdot f)x$

Proof: $[fx, gy] = (fx) \circ (gy) - (gy) \circ (fx)$

$$= f(x \circ g)y + fg(xy)$$
$$- g(y \circ f)x - gf(yx)$$

(Using Leibnitz rule for tangent spaces)

Since $f, g \in C^\infty(M)$, they commute $\Rightarrow fg = gf$

Thus, $[fx, gy] = fg[x, y] + f(x \circ g)y - g(y \circ f)x$

$\llcorner (*)$

$\longrightarrow$

We shall now prove i)

$$T(fx, y) = \nabla_{fx} y - \nabla_y (fx) - [fx, y]$$

$$= \underbrace{f \nabla_x y - (y \circ f) x - f \nabla_y x}_{\text{from properties of Affine connection}} - f[x,y] \underbrace{+ (y \circ f) x}_{\text{By setting } g = 1 \text{ in } (*)}$$

$$= f \{ \nabla_x y - \nabla_y x - [x,y] \} = f T(x,y)$$

$$T(x, gy) = \nabla_x (gy) - \nabla_{gy} (x) - [x, gy]$$

$$= (x \circ g) y + g \nabla_x y - g \nabla_y x - g [x,y] - (x \circ g) y$$

$$= g \{ \nabla_x y - \nabla_y x - [x,y] \}$$

$$= g T(x,y)$$

Since $T(fx, y) = f T(x,y)$ and $T(x, gy) = g T(x,y)$

$T(x,y)$ is $C^\infty(M)$ - linear in the both arguments.

Now we shall prove (ii)

$$R(fx, y)z = \nabla_{fx} \nabla_y z - \nabla_y \nabla_{fx} z - \nabla_{[fx,y]} z$$

$$= f \nabla_x \nabla_y z - \nabla_y (f \underbrace{\nabla_x z}_{\in \mathfrak{X}(M)}) - \nabla_{f[x,y] - (y \circ f) x} z$$

$$= f \nabla_x \nabla_y z - (y \circ f)(\nabla_x z) - f \nabla_y \nabla_x z - \nabla_{f[x,y]} z$$

$$+ \nabla_{(y \circ f) x} z$$

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$$= f \nabla_x \nabla_y z - \cancel{(y \circ f)}(\nabla_x z) - f \nabla_y \nabla_x z - f \nabla_{[x,y]} z$$
$$+ \cancel{(y \circ f)} \nabla_x z$$

$$= f \{ \nabla_x \nabla_y z - \nabla_y \nabla_x z - \nabla_{[x,y]} z \}$$
$$= f R(x,y) z$$

$R(x, gy) z$

$$= \nabla_x \nabla_{gy} z - \nabla_{gy} \nabla_x z - \nabla_{[x,gy]} z$$
$$= \nabla_x (g \underbrace{\nabla_y z}_{\in \mathfrak{X}(M)}) - g \nabla_y \nabla_x z - \nabla_{g[x,y] + (x \circ g) y} z$$
$$= g \nabla_x \nabla_y z + (x \circ g) \nabla_y z - g \nabla_y \nabla_x z - \nabla_{g[x,y]} z - \nabla_{(x \circ g) y} z$$
$$= g \nabla_x \nabla_y z + \cancel{(x \circ g) \nabla_y z} - g \nabla_y \nabla_x z - g \nabla_{[x,y]} z - \cancel{(x \circ g) \nabla_y z}$$
$$= g \{ \nabla_x \nabla_y z - \nabla_y \nabla_x z - \nabla_{[x,y]} z \}$$
$$= g R(x,y) z$$

$R(x,y)(fz)$ $\qquad f \in \mathfrak{X}(M)$

$$= \nabla_x \nabla_y (fz) - \nabla_y \nabla_x (fz) - \nabla_{[x,y]} (fz)$$
$$= \nabla_x (\underbrace{(y \circ f)}_{\in C^\infty(M)} z + f \underbrace{\nabla_y z}_{\in \mathfrak{X}(M)}) - \nabla_y (\underbrace{(x \circ f)}_{\in C^\infty(M)} z + f \nabla_x z)$$
$$- f \nabla_{[x,y]} z - ([x,y] \circ f) z$$

$$= \cancel{(y \circ f)} \nabla_x z + \cancel{x \circ (y \circ f)} z + f \nabla_x \nabla_y z + \cancel{(x \circ f) \nabla_y} z$$
$$- \cancel{y \circ (x \circ f)} z - \cancel{(x \circ f)} \nabla_y z - f \nabla_y \nabla_x z - \cancel{(y \circ f)} \nabla_x z$$
$$- f \nabla_{[x,y]} z - x \cancel{(y \circ f)} z + y \cancel{(x \circ f)} z$$

$$= f\left(\nabla_x \nabla_y z - \nabla_y \nabla_x z - \nabla_{[x,y]} z\right)$$

$$= f\, R(x,y)z$$

Thus we have seen that

$$R(fx,y)z = R(x,fy)z = R(x,y)(fz) = f\,R(x,y)z$$

Thus, $R(x,y)z$ is linear in all 3 arguments.