Proposition 6.3 *Given $X, Y, Z \in \mathfrak{X}(M)$, we have:*

(i) **Bilinearity:** *for any $\alpha, \beta \in \mathbb{R}$,*

$$[\alpha X + \beta Y, Z] = \alpha[X, Z] + \beta[Y, Z]$$
$$[X, \alpha Y + \beta Z] = \alpha[X, Y] + \beta[X, Z];$$

(ii) **Antisymmetry:**

$$[X, Y] = -[Y, X];$$

(iii) **Jacobi identity:**

$$[[X, Y], Z] + [[Y, Z], X] + [[Z, X], Y] = 0;$$

(iv) **Leibniz rule:** *For any $f, g \in C^\infty(M)$,*

$$[f X, g Y] = fg [X, Y] + f(X \cdot g)Y - g(Y \cdot f)X.$$

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Exercise in Lecture 4.

Proof of proposition

(i) Linearity

$[\alpha X + Y, \beta Z + W]$, $X, Y, Z, W \in \mathfrak{X}(M)$, $\beta, \alpha \in \mathbb{R}$

$$(\alpha X + Y)(\beta Z + W) - (\beta Z + W)(\alpha X + Y)$$
$$= \alpha\beta(XZ - ZX) + (YW - WY) + (Y\beta Z - \beta Z Y)$$
$$+ \alpha XW - W\alpha X$$
$$= \alpha\beta[X, Z] + [Y, W] + \beta[Y, Z] + \alpha[X, W]$$

(ii) Antisymmetry:

$$[X, Y] = XY - YX = -(YX - XY) = -[Y, X]$$

(iii) $[[X, Y], Z] = [XY - YX, Z]$ —①

$[[Y, Z], X] = [YZ - ZY, X]$ —②

$[[Z, X], Y] = [ZX - XZ, Y]$ —③

$$① + ② + ③ = [XY, Z] - [YX, Z]$$
$$+ [YZ, X] - [ZY, X]$$
$$+ [ZX, Y] - [XZ, Y]$$

$$+[XY,Z]-[YX,Z]$$
$$+[YZ,X]-[ZY,X]$$
$$+[ZX,Y]-[XZ,Y]$$

$$\cancel{XYZ}-\cancel{ZXY}+\cancel{YXZ}+\cancel{ZYX}$$
$$\cancel{YZX}-\cancel{XYZ}-\cancel{ZYX}+\cancel{XZY}$$
$$\cancel{ZXY}-\cancel{YZX}-\cancel{XZY}+\cancel{YXZ}$$

$$= 0$$

(iv) Leibniz rule:

$$[fX, gY] = (fX)(gY) - (gY)(fX)$$
$$= fX(gY) - gY(fX)$$
$$= f(X\cdot g)Y + fg(XY) - g(Y\cdot f)X - gf(YX)$$
$$= fg[X,Y] + f(X\cdot g)Y - g(Y\cdot f)X$$