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Exercise

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Lemma: If ∇ is torsion free theorem,
Then $R(X,Y)Z + R(Y,Z)X + R(Z,X)Y = 0$

Recall $R(X,Y)Z = \nabla_X \nabla_Y Z - \nabla_Y \nabla_X Z - \nabla_{[X,Y]} Z$

torsion free $: T(X,Y) = \nabla_X Y - \nabla_Y X - [X,Y] = 0 \quad \forall X, Y$

$\iff \nabla_X Y - \nabla_Y X = [X,Y]$

Proof $R(X,Y)Z + R(Y,Z)X + R(Z,X)Y$

$= \nabla_X \nabla_Y Z - \nabla_Y \nabla_X Z - \nabla_{[X,Y]} Z + \nabla_Y \nabla_Z X - \nabla_Z \nabla_Y X - \nabla_{[Y,Z]} X$
$\quad + \nabla_Z \nabla_X Y - \nabla_X \nabla_Z Y - \nabla_{[Z,X]} Y$

$= \nabla_X(\nabla_Y Z - \nabla_Z Y) + \nabla_Y(\nabla_Z X - \nabla_X Z) + \nabla_Z(\nabla_X Y - \nabla_Y X) - \nabla_{[X,Y]} Z - \nabla_{[Y,Z]} X$
$\quad - \nabla_{[Z,X]} Y$

torsion free

$\overset{\text{torsion free}}{=} \nabla_X [Y,Z] + \nabla_Y [Z,X] + \nabla_Z [X,Y] - \nabla_{[X,Y]} Z - \nabla_{[Y,Z]} X - \nabla_{[Z,X]} Y$

$= (\nabla_X [Y,Z] - \nabla_{[Y,Z]} X) + (\nabla_Y [Z,X] - \nabla_{[Z,X]} Y) + (\nabla_Z [X,Y] - \nabla_{[X,Y]} Z)$

$= [X,[Y,Z]] + [Y,[Z,X]] + [Z,[X,Y]] = 0$

torsion free

Jacobi identity