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Differential Geometry Lecture 2 Exercises

Exercise 1: An example of a smooth manifold with an atlas.

$\mathcal{M} := \{\vec{r} := (x, y, z) \in \mathbb{R}^3 \mid x^2+y^2+z^2=1\}$

as a relative topology in the topo. manifold $\mathbb{R}^3$

with the trival definition of open sets.

One atlas of $\mathcal{M}$ contains 2 homeomorphisms φ_α and $\varphi_\beta : \mathbb{R}^2 \mapsto \mathcal{M}$.

Define firstly $\theta := 2\arctan(\sqrt{x^2+y^2})$; Then $\sin\theta = \frac{2\sqrt{x^2+y^2}}{1+x^2+y^2}$, $\cos\theta = \frac{1-x^2-y^2}{1+x^2+y^2}$

$$\varphi_\alpha(x,y) := \left(\frac{x\sin\theta}{\sqrt{x^2+y^2}}, \frac{y\sin\theta}{\sqrt{x^2+y^2}}, \cos\theta\right) = \left(\frac{2x}{1+x^2+y^2}, \frac{2y}{1+x^2+y^2}, \frac{1-x^2-y^2}{1+x^2+y^2}\right)$$

$\Rightarrow \varphi_\alpha(\mathbb{R}^2) = \mathcal{M}\setminus\{(0,0,-1)\} =: \mathcal{M}_\alpha$;

$\varphi_\alpha(x_1,y_1) = \varphi_\alpha(x_2,y_2) \xRightarrow{0\le\theta<\pi} \theta_1 = \theta_2$

$\Rightarrow \frac{\sin\theta_1}{\sqrt{x_1^2+y_1^2}} = \frac{\sin\theta_2}{\sqrt{x_2^2+y_2^2}} \Rightarrow x_1 = x_2, y_1 = y_2$

$\therefore$ Injective $\therefore$ Bijective between $\mathbb{R}^2 \leftrightarrow \mathcal{M}_\alpha$

$$\varphi_\beta(x,y) := \left(\frac{x\sin\theta}{\sqrt{x^2+y^2}}, \frac{y\sin\theta}{\sqrt{x^2+y^2}}, -\cos\theta\right) = \left(\frac{2x}{1+x^2+y^2}, \frac{2y}{1+x^2+y^2}, \frac{-1+x^2+y^2}{1+x^2+y^2}\right)$$

$\Rightarrow \varphi_\beta(\mathbb{R}^2) = \mathcal{M}\setminus\{(0,0,1)\} =: \mathcal{M}_\beta$;

Same argument, Bijective between $\mathbb{R}^2 \leftrightarrow \mathcal{M}_\beta$

$\varphi_\alpha, \varphi_\beta \in C^0(\mathbb{R}^2, \mathbb{R}^3) \Rightarrow \begin{array}{l}\varphi_\alpha, \varphi_\beta \in C^0(\mathbb{R}^2, \mathcal{M}) \\ \varphi_\alpha^{-1}, \varphi_\beta^{-1} \in C^0(\mathcal{M}, \mathbb{R}^2)\end{array}$

$\Rightarrow \therefore \varphi_\alpha, \varphi_\beta$ are homeomorphisms.

Condition 1: $\varphi_\alpha(\mathbb{R}^2) \cup \varphi_\beta(\mathbb{R}^2) = \mathcal{M}$ satisfied.

Condition 2: Let $\tilde{x} := \frac{x}{x^2+y^2}$ and $\tilde{y} := \frac{y}{x^2+y^2}$ for $(x,y) \neq (0,0)$. Then

$\sqrt{\tilde{x}^2+\tilde{y}^2} = (\sqrt{x^2+y^2})^{-1}$

$\tilde{\theta} := 2\arctan(\sqrt{\tilde{x}^2+\tilde{y}^2}) = 2\operatorname{arccot}(\sqrt{x^2+y^2}) = 2\operatorname{arccot}(\tan\frac{\theta}{2}) = \pi - \theta$

$$\Rightarrow \varphi_\alpha(\tilde{x},\tilde{y}) = \left(\frac{\tilde{x}\sin\tilde{\theta}}{\sqrt{\tilde{x}^2+\tilde{y}^2}}, \frac{\tilde{y}\sin\tilde{\theta}}{\sqrt{\tilde{x}^2+\tilde{y}^2}}, \cos\tilde{\theta}\right)$$

$$= \left(\frac{x\sqrt{x^2+y^2}\sin(\pi-\theta)}{x^2+y^2}, \frac{y\sqrt{x^2+y^2}\sin(\pi-\theta)}{x^2+y^2}, \cos(\pi-\theta)\right)$$

$$= \left(\frac{x\sin\theta}{\sqrt{x^2+y^2}}, \frac{y\sin\theta}{\sqrt{x^2+y^2}}, -\cos\theta\right) = \varphi_\beta(x,y) \text{ and}$$

$\varphi_\beta(\tilde{x},\tilde{y}) = \varphi_\alpha(x,y)$ for the same argument

$\Rightarrow \varphi_\alpha^{-1} \circ \varphi_\beta(x,y) = \varphi_\beta^{-1} \circ \varphi_\alpha(x,y) = (\tilde{x},\tilde{y}) = \frac{1}{x^2+y^2}(x,y)$.

We have also

$V_{\alpha\beta} = \mathcal{M}_\alpha \cap \mathcal{M}_\beta = \mathcal{M}\setminus\{(0,0,1),(0,0,-1)\}$

$\Rightarrow \varphi_\alpha^{-1}(V_{\alpha\beta}) = \varphi_\beta^{-1}(V_{\alpha\beta}) = \mathbb{R}^2\setminus\{\mathbf{0}\}$

Since $\frac{1}{x^2+y^2}(x,y)$ is of $C^\infty(\mathbb{R}^2\setminus\{\mathbf{0}\}, \mathbb{R}^2\setminus\{\mathbf{0}\})$ at any point, condition 2 is satisfied.

Then $\mathcal{M}$ is a smooth manifold.

Another example: $M := SO(3)$ as a relative topology in $M_{3\times 3}(\mathbb{R}) \equiv \mathbb{R}^{3\times 3}$, or $\mathbb{R}^9$.

(coordinated by $(-\pi,\pi]\times[0,\pi]\times(-\pi,\pi] =: \widetilde{SO}(3)$)

Set $\psi : \widetilde{SO}(3) \longrightarrow SO(3)$, bijectively

$$\psi\begin{pmatrix}\alpha\\ \beta\\ \theta\end{pmatrix} := \begin{pmatrix} \cos\alpha\sin\beta & \sin\alpha\cos\theta+\cos\alpha\cos\beta\sin\theta & -\sin\alpha\sin\theta+\cos\alpha\cos\beta\cos\theta \\ \sin\alpha\sin\beta & -\cos\alpha\cos\theta+\sin\alpha\cos\beta\sin\theta & \cos\alpha\sin\theta+\sin\alpha\cos\beta\cos\theta \\ \cos\beta & -\sin\beta\sin\theta & -\sin\beta\cos\theta \end{pmatrix}$$

Then for any $(x,y,z) \in \mathbb{R}^3\setminus\{0\}$, let

$$\left.\begin{array}{llll} \text{“}\alpha & = \mathrm{Arg}(x+iy) & \in(-\pi,\pi]\text{”} & \text{(Longitude of the first column)} \\ \beta & = 2\arctan(\sqrt{x^2+y^2}) & \in(0,\pi) & (\tfrac{\pi}{2}\text{- Latitude of the first column)} \\ \theta & = 2\arctan(z) & \in(-\pi,\pi) & \text{(Rotation of the other columns)} \end{array}\right\}\text{in sphere coordinates}$$

And define $\varphi_i : \mathbb{R}^3\setminus\{0\} \longrightarrow SO(3)$, with $r := \sqrt{x^2+y^2}$,

$$\varphi_1\begin{pmatrix}x\\ y\\ z\end{pmatrix} := \begin{pmatrix} \frac{x}{r}\sin\beta & \frac{y}{r}\cos\theta+\frac{x}{r}\cos\beta\sin\theta & -\frac{y}{r}\sin\theta+\frac{x}{r}\cos\beta\cos\theta \\ \frac{y}{r}\sin\beta & -\frac{x}{r}\cos\theta+\frac{y}{r}\cos\beta\sin\theta & \frac{x}{r}\sin\theta+\frac{y}{r}\cos\beta\cos\theta \\ \cos\beta & -\sin\beta\sin\theta & -\sin\beta\cos\theta \end{pmatrix}$$

$$= \begin{pmatrix} \frac{2x}{1+x^2+y^2} & \frac{y(1-z^2)}{r(1+z^2)}+\frac{x(1-x^2-y^2)(2z)}{r(1+x^2+y^2)(1+z^2)} & -\frac{2yz}{r(1+z^2)}+\frac{x(1-x^2-y^2)(1-z^2)}{r(1+x^2+y^2)(1+z^2)} \\ \frac{2y}{1+x^2+y^2} & -\frac{x(1-z^2)}{r(1+z^2)}+\frac{y(1-x^2-y^2)(2z)}{r(1+x^2+y^2)(1+z^2)} & \frac{2xz}{r(1+z^2)}+\frac{y(1-x^2-y^2)(1-z^2)}{r(1+x^2+y^2)(1+z^2)} \\ \frac{1-x^2-y^2}{1+x^2+y^2} & -\frac{4\sqrt{x^2+y^2}\cdot z}{(1+x^2+y^2)(1+z^2)} & -\frac{2\sqrt{x^2+y^2}(1-z^2)}{(1+x^2+y^2)(1+z^2)} \end{pmatrix}$$

$$\varphi_2\begin{pmatrix}x\\ y\\ z\end{pmatrix} := \varphi_1\begin{pmatrix}x\\ y\\ z\end{pmatrix}\underbrace{\begin{pmatrix}0&1&0\\0&0&1\\1&0&0\end{pmatrix}}_{\in SO(3)}; \quad \varphi_3\begin{pmatrix}x\\ y\\ z\end{pmatrix} := \varphi_1\begin{pmatrix}x\\ y\\ z\end{pmatrix}\underbrace{\begin{pmatrix}0&0&1\\1&0&0\\0&1&0\end{pmatrix}}_{\in SO(3)}$$

Condition 1:

$$\mathrm{im}\,\varphi_1 = SO(3)\Big\backslash\left\{\overbrace{\begin{pmatrix}0&a&b\\0&c&d\\\pm1&0&0\end{pmatrix}}^{=:M_1},\ \overbrace{\begin{pmatrix}\frac{x}{r}\sin\beta & -\frac{y}{r} & -\frac{x}{r}\cos\beta\\ \frac{y}{r}\sin\beta & \frac{x}{r} & -\frac{y}{r}\cos\beta\\ \cos\beta & 0 & \sin\beta\end{pmatrix}}^{=:M_2\in SO(3)}\ \middle|\ M_1\in SO(3),\ \begin{pmatrix}x\\y\\z\end{pmatrix}\in\mathbb{R}^3\setminus\{0\}\right\}$$

Since $SO(3) \ni M_1 = \underbrace{\begin{pmatrix}b&0&a\\d&0&c\\0&\pm1&0\end{pmatrix}}_{\in\mathrm{im}\,\varphi_1}\begin{pmatrix}0&0&1\\1&0&0\\0&1&0\end{pmatrix} \in \mathrm{im}\,\varphi_3$,

$$SO(3)\ni M_2 = \underbrace{\begin{pmatrix}-\frac{y}{r} & -\frac{x}{r}\cos\beta & \frac{x}{r}\sin\beta\\ \frac{x}{r} & -\frac{y}{r}\cos\beta & \frac{y}{r}\sin\beta\\ 0 & \sin\beta & \cos\beta\end{pmatrix}}_{\in\mathrm{im}\,\varphi_1}\begin{pmatrix}0&1&0\\0&0&1\\1&0&0\end{pmatrix} \in \mathrm{im}\,\varphi_2,$$

Then $\bigcup_{i=1}^{3}\mathrm{im}\,\varphi_i = SO(3)$. Satisfied.

Condition 2: Obviously $\forall i\in\{1,2,3\}: \varphi_i$ is homeomorphism ($\Rightarrow \varphi_i$ is invertible) and $\forall i,j\in\{1,2,3\}: \varphi_i^{-1}\circ\varphi_j \in C^\infty(\mathbb{R}^3\setminus\{0\}, \mathbb{R}^3\setminus\{0\})$. Satisfied.

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In the example of $SO(3)$, φ_i $(1 \leq i \leq 3)$ are homeomorphisms because

$$\begin{array}{c} \varphi_1(x_1, y_1, z_1) \\ \shortparallel \\ \varphi_1(x_2, y_2, z_2) \end{array} \Rightarrow \begin{cases} \cos\beta_1 = \cos\beta_2 \xrightarrow{0<\beta<\pi} \beta_1 = \beta_2 \Rightarrow r_1 = \tan\frac{\beta_1}{2} = \tan\frac{\beta_2}{2} = r_2 \\ \frac{\sin\beta_1}{r_1}\begin{pmatrix} x_1 \\ y_1 \end{pmatrix} = \frac{\sin\beta_2}{r_2}\begin{pmatrix} x_2 \\ y_2 \end{pmatrix} \xrightarrow[r_1 = r_2]{\beta_1 = \beta_2} x_1 = x_2,\ y_1 = y_2 \\ \left.\begin{array}{l} -\sin\beta_1 \sin\theta_1 = -\sin\beta_2 \sin\theta_2 \Longrightarrow \sin\theta_1 = \sin\theta_2 \\ -\sin\beta_1 \cos\theta_1 = -\sin\beta_2 \cos\theta_2 \xrightarrow{\beta_1 = \beta_2} \cos\theta_1 = \cos\theta_2 \end{array}\right\} \Rightarrow \theta_1 = \theta_2 \Rightarrow z_1 = z_2 \end{cases}$$

Thus φ_1 is injective. $\Rightarrow$ bijective between $\mathbb{R}^3 \setminus \{0\}$ and $\operatorname{im} \varphi_1$

By the same argument, so do φ_2 and φ_3.

Since $\varphi_i \in C^0(\mathbb{R}^3 \setminus \{0\}, \mathbb{R}^{3\times 3})$, $\because$ by relative topology $\varphi_i \in C^0(\mathbb{R}^3 \setminus \{0\}, SO(3))$

$\varphi_i^{-1} \in C^0(SO(3), \mathbb{R}^3 \setminus \{0\})$

Then φ_i $(1 \leq i \leq 3)$ are homeomorphisms.