IV Riemannian Manifolds

IV. 1 Definition and basic properties

Recall that if V is a real vector space of dimension n,

a POSITIVE DEFINATE BILINEAR FORM is a map $\phi : V \times V \mapsto \mathbb{R}$

which is linear in each argument

and s.t. $\phi(v,v) \geq 0 \ \forall v \in V$ and $\phi(v,v) = 0 \Leftrightarrow v = 0$.

ϕ is SYMMETRIC if $\phi(v_1, v_2) = \phi(v_2, v_1)$.

Def. A smooth manifold with a positive definite symmetric bilinear tensor field is called a RIEMANNIAN MANIFOLD.

$\Leftrightarrow \exists \phi \in \mathcal{J}^2(M)$:

$$\phi_p \in \Sigma^2(T_p(M)) \wedge [\forall X_p \in T_p(M) : \phi_p(X_p, X_p) \geq 0 \text{ with } = 0 \Leftrightarrow X_p = 0]$$

We call ϕ a RIEMANNIAN METRIC.

Lemma: If $F : M \mapsto N$ is an IMMERTION $(:\Leftrightarrow \dim F(M) = \dim M$; see App. 2)

and if ϕ is a Riemannian metric on N,

Then $F^*(\phi) \in \mathcal{J}^2(M)$ is a Riemannian metric on M.

Proof as exercise ; recall that

$$(F^*\phi)(X_p, Y_p) := \phi(\overset{\in T_p(N)}{F_*(X_p)}, \underbrace{F_*(Y_p)}_{= 0 \text{ iff } Y_p = 0})$$

Thm. Any smooth manifold can be endowed with a Riemannian metric.

"2 proofs" : ① Use a covering + local coordinate system + Lemma above

② Use Whitney Imbedding Thm + Lemma above

Remark: For a Riemannian manifold, $T_p(M)$ has an inner product provided by ϕ

$\Rightarrow$ We can now define orthonormal bases on $T_p(M)$ at every $p \in M$.

Thm. Let (M, ϕ) be a Riemannian manifold which is oriented.

$\hookrightarrow$ fix ϕ from now on

Then $\exists !$ volume form Ω s.t. $\forall p \in M : \Omega_p(F_{1,p}, \cdots, F_{n,p}) = 1$ (*)

whenever $\{F_{1,p}, \cdots, F_{n,p}\}$ is an oriented orthonormal basis of $T_p(M)$.

Proof : Since $\dim(\Lambda^n(T_p(M))) = 1$, then Ω is uniquely defined by (*).

We have to show that it does not vanish.

Let (U, φ) be a local chart with $p \in U$;

Let $\{E_{1,p}, \cdots, E_{n,p}\}$ be the corresponding basis for $T_p(M)$. (Coordinate frame at p)

Set $g_{ij}(p) := \phi_p(E_{i,p}, E_{j,p})$.

Since $E_{i,p} = \sum_{k=1}^n \alpha_i^k F_{k,p}$ and since $\phi_p(F_{i,p}, F_{j,p}) = \delta_{ij}$

$\Rightarrow$ $g_{i,j}(p) = \phi_p(E_{i,p}, E_{j,p}) = \phi_p\left(\sum_{k=1}^n \alpha_i^k F_{k,p}, \sum_{l=1}^n \alpha_j^l F_{k,p}\right)$

$= \sum_{k=1}^n \alpha_i^k \alpha_j^k = ({}^T A A)_{ij}$ with $A_{ij} = \alpha_j^i$

$\Rightarrow$ $\det(g_{ij}(p))_{ij} = \det({}^T A A) = (\det(A))^2 > 0$

$\Rightarrow$ $\sqrt{\det(g_{ij}(p))_{ij}} > 0$

$\Rightarrow$ $\Omega_p(E_{1,p}, \cdots, E_{n,p}) \overset{\text{exercise}}{=} \det(A) \underbrace{\Omega_p(F_{1,p}, \cdots, F_{n,p})}_{=1 \text{ by def}} = \det(A) = \sqrt{\det(g_{ij})} > 0$ (>0 by choice of orientation of $(F_{1,p}, \cdots, F_{n,p})$)

Since p, (U, φ) are arbitrary, then Ω is a volume form.

Smoothness is automatic. $\square$

Ω is called the NATURAL VOLUME ELEMENT on the oriented Riemannian manifold (M, ϕ).

We often see $\underbrace{\varphi^* \Omega}_{\in \Lambda^n(\mathbb{R}^n)} = \sqrt{g}\, dx_1 \wedge \cdots \wedge dx_n$

$\hookrightarrow g := \det(g_{ij} \circ \varphi^{-1})$

Remark: We can use Ω to define

$\int_M f := \int_M f\Omega \quad \forall f \in C^\infty(M)$

Let $c: [a,b] \mapsto M$ be a smooth curve on a Riemannian manifold (M, ϕ).

The tangent vector is

$c_*\left(\frac{d}{dt}\Big|_t\right) =: \dot{c}(t) \in T_{c(t)}(M)$

Def. The LENGTH of the curve is defined by

$L := \int_a^b [\phi_{c(t)}(\dot{c}(t), \dot{c}(t))]^{\frac{1}{2}} dt$

Exercise: This is indep. of the parametrization.

The ARC LENGTH is defined by $s: [a,b] \mapsto [0, L]$,

$s(t) := \int_a^t [\phi_{c(\tau)}(\dot{c}(\tau), \dot{c}(\tau))]^{\frac{1}{2}} d\tau$

We often write $\left[\left(\frac{ds}{dt}\right)^2 = \phi(\dot{c}, \dot{c})\right]$

Thm. [Bo. p. 189~191] A connected manifold is a metric space with the metric defined by $d(p,q) = \inf$ on the length of all paths (:= curves of C^1 or C^∞) between p and q.

The metric topology and the manifold topology coincide.

Reminder: a METRIC SPACE is a pair (M, d) with $d: M \times M \mapsto \mathbb{R}_+$ s.t.

1) $d(x,y) \geq 0$

2) $d(x,y) = 0 \Leftrightarrow x = y$

3) $d(x,y) = d(y,x)$

4) $d(x,z) \leq d(x,y) + d(y,z)$ ($\triangle$ inequality)