III. Integration on manifolds

III.1 Integration of n-forms

Let $\mathcal{M}$ be an oriented $\Lambda^n(\mathcal{M})$ manifold and let $\{(U_j, \varphi_j)\}_j$ be an oriented $\overbrace{\quad}^{\text{preserving}}$ atlas.

Let $\omega \in \Lambda^n(\mathcal{M})$ with $\mathrm{supp}(\omega) \subset U_j$ and with $\mathrm{supp}(\omega)$ compact.

$\Rightarrow \overset{(\text{or } \omega_p)}{\omega(p)} = a(p)(dx^1)_p \wedge \cdots \wedge (dx^n)_p$ with $a \in C^\infty(\mathcal{M})$

Recall that $\varphi_j^{-1\,*}$ maps $\Lambda^n(\mathcal{M})$ to $\Lambda^n(\mathbb{R}^n)$

$\Rightarrow \varphi_j^{-1\,*}(\omega) = \underbrace{a \circ \varphi_j^{-1}}_{\text{function on } \varphi_j(U_j) \subset \mathbb{R}^n} dx^1 \wedge \cdots \wedge dx^n.$

Then we set

$$\int_{\mathcal{M}} \omega = \int_{U_j} \omega := \overset{\text{usual Riemann integral in } \mathbb{R}^n}{\int_{\varphi_j(U_j)}} a \circ \varphi_j^{-1}\, dx_1 \cdots dx_n \equiv \int_{\varphi_j(U_j)} a(x)\, dV \qquad (*)$$

Lemma: If $\mathrm{supp}(\omega) \subset U_k$ for an other localization map (U_k, φ_k), then

$$\int_{\varphi_k(U_k)} a \circ \varphi_k^{-1}\, dx_1 \cdots dx_n = \int_{\varphi_j(U_j)} a \circ \varphi_j^{-1}\, dx_1 \cdots dx_n$$

(independence of the coordinate system) (proof as Exercise)

Def. Let $\mathcal{M}$ be an oriented smooth manifold, $\{(U_j, \varphi_j)\}_j$ a covering preserving the orientation, and $\omega \in \Lambda^n(\mathcal{M})$ with compact support. $:\Leftrightarrow \forall j : \mathrm{supp}(f_j) = U_j$

Let $\{f_j\}$ be a partition of unity of $\mathcal{M}$ subordinated to U_j. Then

$$\int_{\mathcal{M}} \omega = \int_{\mathcal{M}} \underset{\uparrow}{\sum_j} f_j \omega = \underset{\uparrow}{\sum_j} \int_{\mathcal{M}} \underbrace{f_j \omega}_{\in \Lambda^n(\mathcal{M})} = \sum_j \int_{U_j} f_j \omega \text{ as defind in } (*).$$

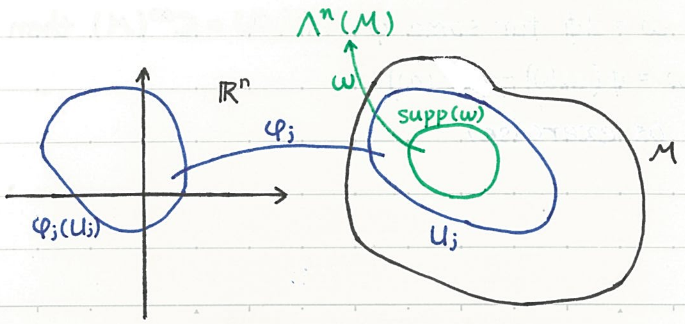
the sum is finite because supp(ω) is compact

Remarks

- $\int_{\mathcal{M}} \omega$ is independent of the choice of a partition of unity. (Exercise)
- The map $\Lambda^n(\mathcal{M}) \ni \omega \mapsto \int_{\mathcal{M}} \omega \in \mathbb{R}$ is a linear map.
- We can avoid the "comapactedly supported" but be careful about the convergence.
- If $F : \mathcal{M} \mapsto \mathcal{N}$ is a diffeomorphism and if $\omega \in \Lambda^n(\mathcal{N})$, compactedly supported,

$$\int_{\mathcal{M}} \underbrace{F^* \omega}_{\in \Lambda^n(\mathcal{M})} = \pm \int_{\mathcal{N}} \omega$$

(± depends on if F preserves the orientation or not)



Thm. (Stokes' Theorem) (The main thm of this chapter)

Let M be an oriented smooth manifold of dim n, with boundary ∂M. (with induced orientation).

Let $i : \partial M \mapsto M$ be the inclusion map. (identity) $\Rightarrow i^* : \Lambda^{n-1}(M) \mapsto \Lambda^{n-1}(\partial M)$

Let $\omega \in \Lambda^{n-1}(M)$ with compact support. Then

$$\int_{\partial M} \underbrace{i^*\omega}_{\in \Lambda^{n-1}(\partial M)} = \int_M \underbrace{d\omega}_{\in \Lambda^n(M)}$$

Reference for the proof: [GN p. 82-84] [Bo p. 260-261]

Remark: 1) If $\partial M = \emptyset$ then $\int_M d\omega = 0$

2) The proof is similar to the one of Calculus II on $\mathbb{R}^2$ or $\mathbb{R}^3$, and the main ingredient is $\int_a^b f'(x)\,dx = f(b) - f(a)$.

Exercise: Show that the Green Thm, Stokes Thm in $\mathbb{R}^3$ or Divergence Thm are special cases of this theorem. See Bo p. 262-263.

Recall that M is orientable iff $\exists \phi \in \Lambda^n(M)\ \forall p \in M : \phi_p \neq 0$.

Def. Let us fix one of them, and for any $f \in C^\infty(M)$ with compact support we set

$\int_M f := \int_M f\phi$ ⚠ This def depends on the choice of ϕ.

In particular if M is compact we set the volume of M as

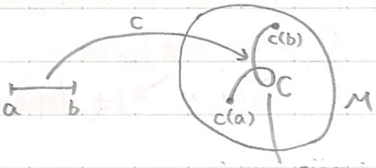
$$\mathrm{Vol}(M) := \int_M 1\phi = \int_M \phi$$

III.2 Line integrals

Let $c : [a,b] \mapsto M$ be a diffeomorphism and set $C = c([a,b])$ ↗Capital ; it is a manifold

If $\omega \in \Lambda^1(M)$ we set

$$\int_C \omega = \int_{[a,b]} \underbrace{c^*\omega}_{\substack{\in \Lambda^1([a,b]) \\ \text{\rotatebox{90}{$\in$}} \\ t \mapsto f(t)\,dt}} = \int_a^b f(t)\,dt$$



an immersion in App. 2

Lemma: If $\omega = d\phi$ for some $\phi \in \Lambda^0(M) = C^\infty(M)$ then

$$\int_C \omega = \phi(c(b)) - \phi(c(a))$$

(Proof as exercise)