Main result of this chapter (for def of grad, div, curl, etc)

Thm. Let M be a smooth manifold, and $\Lambda(M)$ the exterior algebra,
There is a unique linear map
$d : \Lambda(M) \mapsto \Lambda(M)$ satisfying $d(\underset{\uparrow}{f})$ this map, differential of f

1) If $f \in \Lambda^0(M) = C^\infty(M)$, then $df = df \in \mathcal{T}^1_0(M)$, $(df)_p(X_p) = X_p(f)$

2) If $\phi \in \Lambda^r(M)$ and $\psi \in \Lambda^s(M)$, then
$$d(\phi \wedge \psi) = (d\phi) \wedge \psi + (-1)^r \phi \wedge (d\psi)$$

3) $d^2 = d \circ d = 0$

In local coordinates, we have an explicit formula for d:
Recall that if (U, φ) is a chart, $p \in U$, then
$\{E_{j,p}\}_{j=1}^n$ is a basis for $T_p(M)$ and $\{(dx^j)_p\}_{j=1}^n$ is a basis for $T_p^*(M)$.
Then $\phi \in \Lambda^r(M)$ can be represented by
$$\phi_p = \sum_{1 \le i_1 < \cdots < i_r \le n} a_{i_1, \ldots, i_r}(p)(dx^{i_1})_p \wedge \cdots \wedge (dx^{i_r})_p \quad \text{(a special case of } \mathcal{T}^0_s(M))$$
$$=: \sum_I a_I(p)(dx^I)_p \quad \text{with } a_I : U \mapsto \mathbb{R} \text{ smooth.}$$
Then (def)
$$(d\phi)_p := \sum_I \overbrace{(da_I)_p}^{\in \mathcal{T}^1_0(M)} \wedge \overbrace{(dx^I)_p}^{\in \Lambda^r(M)} \in \Lambda^{r+1}(M)$$

Exercise: check that this def satisfies the 3 conditions [GN p74]

Remarks
1) d is a <u>local</u> operator: If $U \subset M$ and $\phi \in \Lambda(U) \subset \Lambda(M)$ then $d_U \phi = d_M \phi$
2) d maps $\Lambda^r(M)$ to $\Lambda^{r+1}(M)$
3) d is called the EXTERIOR DERIVATIVE

Exercise (Thm?)
If $\omega \in \Lambda^1(M)$ and $X, Y \in \mathfrak{X}(M) := \{C^\infty\text{-vector fields}\}$, then
$$d\omega(X, Y) = X\underbrace{\omega(Y)}_{\in C^\infty(M)} - Y\underbrace{\omega(X)}_{\in C^\infty(M)} - \underbrace{\omega([X, Y])}_{\in C^\infty(M)} \in C^\infty(M)$$

$$(\omega X)_p = \omega_p(X_p) \in \mathbb{R}$$

Proof: In a chart (U, φ), $\omega_p = \sum_{j=1}^n a_j(p)(dx^j)_p$
For shortness, we write $\omega_p = f\,dg$ for $f, g \in C^\infty(M)$
Then $d\omega(X, Y) = d(f dg)(X, Y) \overset{\text{by def}}{=\!=} (df \wedge dg)(X, Y) \overset{\text{by def}}{=\!=} df(X)dg(Y) - df(Y)dg(X)$
$$= (Xf)(Yg) - (Yf)(Xg) \in C^\infty(M)$$

$$X\omega(Y) - Y\omega(X) - \omega([X, Y]) = X(f dg(Y)) - Y(f dg(X)) - f dg([X, Y])$$
$$= X(fYg) - Y(fXg) - f(XY - YX)g \overset{\text{Leibniz}}{=\!=} XfYg + fXYg - YfXg - fYXg - fXYg + fYXg$$
$$= (Xf)(Yg) - (Yf)(Xg) \quad \square$$

$$X\omega(Y) - Y\omega(X) - \omega([X,Y]) = X(fdg(Y)) - Y(fdg(X)) - fdg([X,Y])$$
$$= X(fYg) - Y(fXg) - f(XY - YX)g \overset{\text{Leibniz}}{=\!=\!=} XfYg + fXYg - YfXg - fYXg - fXYg + fYXg$$
$$= (Xf)(Yg) - (Yf)(Xg)$$ □

Generalization

Prop. ["GN" 3.8.2 p.75~76] (independent of any coordinate systems)

Let $\phi \in \Lambda^r(M)$ and $X_1, \cdots, X_{n+1} \in \mathfrak{X}(M)$, then

omit

$$[d\phi](X_1, \cdots, X_{r+1}) = \sum_{i=1}^{n+1} (-1)^{r+1} X_i \phi(X_1, \cdots, \hat{X}_i, \cdots, X_{r+1})$$
$$+ \sum_{i<j} (-1)^{i+j} \underbrace{\phi([X_i, X_j], X_1, \cdots, \hat{X}_i, \cdots, \hat{X}_j, \cdots, X_{r+1})}_{\in C^\infty(M)},$$

Exercise: This satisfies conditions 1~3 of Thm.

Recall that if $F: M \mapsto N$ a smooth map between manifolds, then

$F^*: \mathcal{T}^r(N) \mapsto \mathcal{T}^r(M)$ by

$$(F^*\phi)_{\underset{\uparrow \in M}{p}} (X_{1,p}, \cdots, \underbrace{X_{r,p}}_{\in T_p(M)}) := \phi_{F(p)} (\underbrace{F_*(X_{1,p})}_{\in T_{F(p)}(N)}, \cdots, F_*(X_{r,p}))$$

which is also $F^*: \Lambda^r(N) \mapsto \Lambda^r(M)$ (alternating property is preserved)

Lemma: In this framework

$F^* \circ d_N = d_M \circ F^*$ [Bo Thm 8.2 p.223]

Exercise for mathematicians: about de Rham cohomology

[GN, p.76 ex 5]

II.4 Orientation on a manifold (easier)

Let V be a real vector space of dim n, and $\{E_j\}_{j=1}^n$ and $\{F_j\}_{j=1}^n$ 2 bases

Set $A \in M_{n\times n}(\mathbb{R})$ by $F_i = \sum_{j=1}^n a_{ij} E_j$ coeff. of the change of basis

Def. The two bases has the SAME ORIENTATION if $\det(A) > 0$

and of OPPOSITE ORIENTATION if $\det(A) < 0$

$\Rightarrow$ There exist 2 classes of equivalence of bases.

We say either they are either POSITIVELY ORIENTED

or NEGATIVELY ORIENTED.

Def. Let M be a smooth manifold of dim $n \geq 1$, ⚠ Convention changed

M is ORIENTABLE if there exists a covering ($\equiv$ atlas) $\{(U_i, \varphi_i)\}_i$ s.t. all transition maps $\varphi_j \circ \varphi_i^{-1} : \varphi_i(U_i \cap U_j) \mapsto \varphi_j(U_i \cap U_j)$ is ORIENTATION PRESERVING

$:\Leftrightarrow$ if $\det \mathrm{Jac}(\varphi_j \circ \varphi_i^{-1}) > 0$

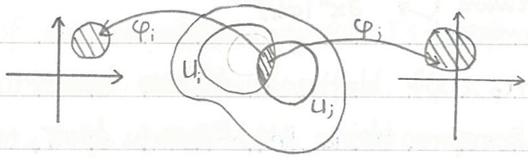
Lemma: A connected orientable manifold of dim ≥ 1 has only 2 possible orientations.

Remark: If $M = \{p\}$ (of dim 0)

an orientation is a map from p to ± 1. We need this because

$$\underset{\uparrow}{\int_a^b} f'(x)dx = \underset{\uparrow}{f(b)} - \underset{\uparrow}{f(a)}$$

M of dim 0, $b \mapsto +1$, $a \mapsto -1$ are what we need from orientations.



Thm. [Bo. p.218] (very deep but intruisive) ϕ is called a VOLUME FORM

A manifold is orientable iff $\exists \phi \in \Lambda^n(M)\ \forall p \in M : \phi_p \neq 0$ $(\phi_p \in \Lambda^n(T_p(M)))$

Recall that $H^n = \{(x_1, \cdots, x_n) \in \mathbb{R}^n \mid x_n \geq 0\}$ and

M is a smooth manifold with boundary if every chart $(U_\alpha, \varphi_\alpha)$ with $\varphi_\alpha : U_\alpha \mapsto H^n$ is a homeomorphism. (+ atlas conditions)

The BOUNDARY of M is denoted by ∂M and is given by

$$\partial M := \bigcup_\alpha \varphi_\alpha^{-1}(\partial H_n \cap \varphi_\alpha(U_\alpha))$$

which is a smooth manifold with dim $(n-1)$

Next time: If M is oriented then it induces also an orientation on ∂M (needed in Stroke's Thm)