If $\mathcal{M} = \mathbb{R}^n$

then φ = identity, and if $f \in C^\infty(p)$ then $(df)_p = \sum_{j=1}^{n} \lambda_i (dx^i)_p$ with (↙ coef on a basis)

$\lambda_i = E_{p,i}(f) = \frac{\partial f}{\partial x^i}(p)$

$\Rightarrow (df)_p = \sum_{i=1}^{n} \frac{\partial f}{\partial x^i}(p)(dx^i)_p$

Corresponds to $df = \frac{\partial f}{\partial x^1} dx^1 + \frac{\partial f}{\partial x^2} dx^2 + \cdots + \frac{\partial f}{\partial x^n} dx^n$, seen in Calculus II.

II.2 Tensor field

Recall that a vector field is a map

$X : \mathcal{M} \mapsto \bigcup_{p \in \mathcal{M}} T_p(\mathcal{M}) \equiv T(\mathcal{M})$.

Def. a (r,s)-TENSOR FIELD on $\mathcal{M}$ is a map

$\phi : \mathcal{M} \mapsto \bigcup_{p\in\mathcal{M}} \mathcal{T}^r_s(T_p(\mathcal{M}))$

$p \mapsto \phi(p) \in \mathcal{T}^r_s(T_p(\mathcal{M}))$ ($T_p(\mathcal{M})$ of dimension n if $\dim \mathcal{M} = n$)

Examples

1) A vector field $X : \mathcal{M} \mapsto T(\mathcal{M})$ is a (0,1)-tensor field. Indeed:

a (0,1)-tensor field ϕ is a map

$\phi : \mathcal{M} \mapsto \bigcup_{p\in\mathcal{M}} \underbrace{\mathcal{T}^0_1(T_p(\mathcal{M}))}$

linear map from $T_p^*(\mathcal{M})$ to $\mathbb{R} \Rightarrow$ element of $T_p^{**}(\mathcal{M}) = T_p(\mathcal{M})$ (↑ an exercise)

2) Reciprocally, a (1,0)-tensor field ϕ is a map

$\phi : \mathcal{M} \mapsto \bigcup_{p\in\mathcal{M}} \underbrace{\mathcal{T}^1_0(T_p(\mathcal{M}))} = \bigcup_{p\in\mathcal{M}} T_p^*(\mathcal{M})$

linear map from $T_p(\mathcal{M})$ to $\mathbb{R} \Rightarrow$ element of $T_p^*(\mathcal{M})$

$\bigcup_{p\in\mathcal{M}} T_p^*(\mathcal{M})$ is called a COTANGENT BUNDLE. (exercise: it's a smooth manifold)

In this case ϕ is a COVECTOR FIELD.

3) A map $\phi : \mathcal{M} \mapsto \bigcup_{p\in\mathcal{M}} \mathcal{T}^2_0(T_p(\mathcal{M}))$ is called FIELD of BILINEAR FORMS.

$\forall p \in \mathcal{M} : \phi_p : T_p(\mathcal{M}) \times T_p(\mathcal{M}) \xrightarrow{\text{bilinear}} \mathbb{R}$.

Observation: A bilinear map can be identified with a $n \times n$ matrix:

$\alpha_{ij,p} := \phi_p(E_{i,p}, E_{j,p}) \quad (i,j \in \{1,\cdots,n\})$

About smoothness

There are several equivalent defs for the smoothness for a tensor field.

For example, if $X_1, \cdots, X_r \in \mathfrak{X}(\mathcal{M}) = \{\text{smooth vector fields}\}$

and if $Y_1, \cdots, Y_s$ are smooth covector fields,

then one imposes that the map

$\mathcal{M} \ni p \mapsto \phi_p(X_{1,p}, \cdots, X_{r,p}, Y_{1,p}, \cdots, Y_{s,p}) \in \mathbb{R}$ is smooth.

Or, if (U, φ) is a chart, if $p \in U$ and if we consider $\{E_{j,p}\}_{j=1}^n$ and $\{(dx^i)_p\}_{i=1}^n$ the coordinate frames and coframes. Then we can write

$$\phi_p = \sum_{\substack{i_k=1 \\ j_\ell=1}}^{n} \underbrace{a_{i_1, \cdots, i_r}^{j_1, \cdots, j_s}(p)}_{\in \mathbb{R}} (dx^{i_1})_p \otimes \cdots \otimes (dx^{i_r})_p \otimes E_{j_1,p} \otimes \cdots \otimes E_{j_s,p}$$

(coefficient in a local basis)

and impose that the coefficients are C^∞ on U.

We call such smooth tensors C^∞-TENSOR FIELDS.

Def. The set of all smooth (r,s)-tensor fields on $\mathcal{M}$ is denoted by $\mathcal{T}_s^r(\mathcal{M})$.

Lemma: 1) $\mathcal{T}_s^r(\mathcal{M})$ is a vector field

2) $\mathcal{T}_0^r(\mathcal{M})$ is a $C^\infty(\mathcal{M})$-module $:\Leftrightarrow \phi(x_1, \cdots, f x_j, \cdots, x_n) = f\phi(x_1, \cdots, x_j, \cdots, x_n)$

3) If $\phi \in \mathcal{T}_0^r(\mathcal{M})$ and $\psi \in \mathcal{T}_0^s(\mathcal{M})$ then $\phi \otimes \psi \in \mathcal{T}_0^{r+s}(\mathcal{M})$

with $(\phi \otimes \psi)_p := \phi_p \otimes \psi_p$

Remarks

1) If $f \in C^\infty(\mathcal{M}) \equiv C^\infty(\mathcal{M}, \mathbb{R})$ then we define a covector field by the formula

$df : \mathcal{M} \mapsto \mathcal{T}^*(\mathcal{M}) = \bigcup_{p \in \mathcal{M}} T_p^*(\mathcal{M})$, $(\Leftrightarrow df \in \mathcal{T}_0^1(\mathcal{M}))$

$(df)_p(X_p) := X_p(f)$

↖ called the DIFFERENTIAL of f

2) If $F : \mathcal{M} \mapsto \mathcal{N}$ a smooth map and if ϕ is a $(r,0)$-tensor field on $\mathcal{N}$ then we set $F^*\phi$ a $(r,0)$-tensor field on $\mathcal{M}$ by

$$(F^*\phi)_p(X_{1,p}, \cdots, X_{r,p}) := \phi_{F(p)}(\underbrace{F_*(X_{1,p})}_{\in T_{F(p)}(\mathcal{N})}, \cdots, F_*(X_{r,p}))$$

It means

$F^* : \mathcal{T}_0^r(\mathcal{N}) \longmapsto \mathcal{T}_0^r(\mathcal{M})$

Def. A tensor field $\phi \in \mathcal{T}^r_0(M)$ is SYMMETRIC if $\forall p \in M: \phi_p \in \Sigma^r(T_p(M))$ → {sym. tensors}
ALTERNATING if Λ → {alt. tensors}

Remark: (Very important)

A symmetric tensor field $\phi \in \mathcal{T}^2_0(M)$ (bilinear forms on M) is POSITIVE DEFINITE if
$\forall p \in M \, \forall X_p \in T_p(M): \phi_p(X_p, X_p) \geq 0$; equality $\Leftrightarrow X_p = 0$

A manifold with a symmetric positive definite bilinear form is called a RIEMANN MANIFOLD ; ϕ is called a RIEMANN METRIC. ($\Rightarrow$ Integration)

(Good for geometry)

II.3 Differential forms and exterior derivative

Def. A tensor field $\phi \in \mathcal{T}^r(M)$ which is alternating is called an EXTERIOR DIFFERENTIAL FORM of degree r ; or a r-FORM.

We write $\Lambda^r(M)$ for the set of all r-forms, and

$\Lambda(M) := \bigoplus_{r=0}^{n} \Lambda^r(M)$, with $\Lambda^0(M) := C^\infty(M)$.

Properties

1) If $\phi \in \Lambda^r(M)$ and $\psi \in \Lambda^s(M)$ then $\phi \wedge \psi \in \Lambda^{r+s}(M)$ [$\phi \wedge \psi = (-1)^{rs}\psi \wedge \phi$]
2) $\Lambda(M)$ is an algebra with the Wedge product $\wedge$.
3) If (U, φ) is a local chart, and if $p \in U$, then the set
$\{(dx^{i_1})_p \wedge \cdots \wedge (dx^{i_r})_p\}$ with $1 \leq i_1 < \cdots < i_r \leq n$
is a basis for $\Lambda^r(T_p(M))$, and accordingly
$\{(dx^{i_1}), \cdots, (dx^{i_r})\}$ is a basis for $\Lambda^r(U) \subset \Lambda^r(M)$.

$\Rightarrow$ $\Lambda(M)$ is the algebra of differential forms or exterior algebra.