II: Tensors, tensor fields and differential forms

II.1 Tensors [Bo 199-214][GN 62-69]

Let V be a finite dimensional and real vector space $(\mathbb{R}^n)$ and let V^* be its DUAL.

(= the set of all linear maps $V \mapsto \mathbb{R}$, such a map is called a LINEAR FUNCTIONAL on V)

Prop. If $\dim V = n$, then $\dim V^* \overset{\text{exercise}}{=} n$

Def. a TENSOR ϕ on V is a multilinear map

$$\phi : \underbrace{V \times V \times V \times \cdots \times V}_{r \text{ terms}} \times \underbrace{V^* \times V^* \times \cdots \times V^*}_{s \text{ terms}} \longmapsto \mathbb{R}$$

e.g.

$$\phi(\underbrace{v_1, \alpha v_2 + \beta v_2'}_{\in V \times V}, \underbrace{\omega_1}_{\in V^*}) = \alpha \phi(v_1, v_2, \omega_1) + \beta \phi(v_1, v_2', \omega_1)$$

We say that ϕ is r-times COVARIANT and s-times CONTRAVARIANT.

We write $\quad \phi \in \mathcal{T}^r_s(V) \equiv \mathcal{T}^{r,s}(V, V^*)$

Examples

1) $r=1, s=0$: $\phi : V \mapsto \mathbb{R}$ is an element of $V^* = \mathcal{T}^1_0(V)$

2) $r=1, s=1$: $\phi(v, \omega) \equiv \omega(v) \equiv \langle \omega, v \rangle$ related to scalar product

Lemma: $\mathcal{T}^r_s(V)$ is a vector space of dim n^{r+s}. (exercise)

Remark: If $\phi_j \in \mathcal{T}^{r_j}_0(V)$, $j = 1, 2$, we set

$\phi_1 \otimes \phi_2 \in \mathcal{T}^{r_1+r_2}_0(V)$ with

$$\phi_1 \otimes \phi_2(v_1, \cdots, v_{r_1}, v_{r_1+1}, \cdots, v_{r_1+r_2}) := \underbrace{\phi_1(v_1, \cdots, v_{r_1})}_{\in \mathbb{R}} \underbrace{\phi_2(v_{r_1+1}, \cdots, v_{r_1+r_2})}_{\in \mathbb{R}}$$

Similar def for $\phi_j \in \mathcal{T}^0_{s_j}(V)$, $j = 1, 2$

If $\phi_1 \in \mathcal{T}^r_0(V) =: \mathcal{T}^r(V)$, $\phi_2 \in \mathcal{T}^0_s(V) =: \mathcal{T}_s(V)$, then

$\phi_1 \otimes \phi_2 \in \mathcal{T}^r_s(V)$ with

$$\phi_1 \otimes \phi_2(v_1, \cdots, v_r, \omega_1, \cdots, \omega_s) := \phi_1(v_1, \cdots, v_2)\, \phi_2(\omega_1, \cdots, \omega_2)$$

⚠ This product is not commutative!

Def. A tensor $\phi \in \mathcal{T}^r(V)$ is SYMMETRIC if invariant under the permutation of 2 argumen

(for example if $\phi(v_1, v_2) = \phi(v_2, v_1)$)

and ALTERNATING if it changes the sign under the permutation of 2 arguments.

(for example if $\phi(v_1, v_2) = -\phi(v_2, v_1)$)

Same def for $\phi \in \mathcal{T}_s(V)$.

We write $\Sigma^r(V)$ for the set of symmetric tensors in $\mathcal{T}^r(V)$
and $\Lambda^r(V)$ for " alternating " $\mathcal{T}^r(V)$.

Note that $\Sigma^r(V)$ and $\Lambda^r(V)$ are vector spaces.

Let S_k denote the group of all permutation of $\{1,\cdots,k\}$

$\sigma \in S_k$ if σ is a bijective map from $\{1,\cdots,k\}$ to itself
with $(1,\cdots,k) \longmapsto (\sigma(1),\cdots,\sigma(k))$

We set $\operatorname{sgn}(\sigma)=1$ if σ corresponds to an even number of transposition,
and $\operatorname{sgn}(\sigma)=-1$ if " odd " .

↳ permutation of 2 element

Def. On $\mathcal{T}^n(V)$ one set

$\mathcal{S}: \mathcal{T}^n(V) \mapsto \mathcal{T}^n(V)$ and $\mathcal{A}: \mathcal{T}^n(V) \mapsto \mathcal{T}^n(V)$ by

$$\left.\begin{aligned}
[\mathcal{S}\phi](v_1,\cdots,v_n) &:= \frac{1}{n!}\sum_{\sigma\in S_n} \phi(v_{\sigma(1)},\cdots,v_{\sigma(n)}) \quad \text{(SYMMETRIZE)}\\
[\mathcal{A}\phi](v_1,\cdots,v_n) &:= \frac{1}{n!}\sum_{\sigma\in S_n} \operatorname{sgn}(\sigma)\phi(v_{\sigma(1)},\cdots,v_{\sigma(n)}) \quad \text{(ANTI-SYMMETRIZE)}
\end{aligned}\right\}\text{linear}$$

Lemma: 1) $\mathcal{S}^2=\mathcal{S}$, $\mathcal{A}^2=\mathcal{A}$

2) $\mathcal{S}\mathcal{T}^n(V)=\Sigma^n(V)$, $\mathcal{A}\mathcal{T}^n(V)=\Lambda^n(V)$

3) $\phi\in\Sigma^n(V)$ iff $\mathcal{S}\phi=\phi$, $\phi\in\Lambda^n(V)$ iff $\mathcal{A}\phi=\phi$.

(iff: if and only if)

Remark: If $F: V\mapsto W$ is a linear map between 2 vector spaces
then it induces a linear map

$F^*: \mathcal{T}^n(W) \mapsto \mathcal{T}^n(V)$ by

$$\underbrace{[F^*\phi]}_{\in\mathcal{T}^n(V)}(\underbrace{v_1,\cdots,v_n}_{\in V^n}) = \underbrace{\phi}_{\in\mathcal{T}^n(W)}(\underbrace{F(v_1),\cdots,F(v_n)}_{\in W^n}) \quad \forall\phi\in\mathcal{T}^n(W)$$

Now let us set

$$\Lambda(V) := \underbrace{\Lambda^0(V)}_{=\mathbb{R}} \oplus \Lambda^1(V) \oplus \Lambda^2(V) \oplus \cdots \oplus \Lambda^j(V) \oplus \cdots$$

$$\subset \mathcal{T}^0(V) \oplus \mathcal{T}^1(V) \oplus \mathcal{T}^2(V) \oplus \cdots \oplus \mathcal{T}^j(V) \oplus \cdots =: \mathcal{T}(V) \leftarrow \text{lemoi algebra over } V$$

The elements of $\Lambda(V)$ or $\mathcal{T}(V)$ cousint in finite "sums" ← only a notation

$$\phi^0+\phi^1+\phi^2+\cdots+\phi^j+\cdots \equiv (\phi^0,\phi^1,\phi^2,\cdots,\phi^j,\cdots) =: \Phi$$

for $\phi^j\in\Lambda^j(V)$ or $\mathcal{T}^j(V)$; $\exists k\in\mathbb{N}\ \forall j\geq k: \phi^j=0$ (k different for each Φ)

Lemma: $\mathcal{T}(V)$ is a vector space and an associative algebra with $\otimes$ ← extended by linearity
↳ $(\phi\otimes\psi)\otimes\varphi = \phi\otimes(\psi\otimes\varphi)$

E.g. $(\phi_0+\phi_1)\otimes(\psi_0+\psi_1+\psi_2)$

$$\begin{aligned}
&= \underbrace{\phi_0\otimes\psi_0}_{\in\mathbb{R}} + \boxed{\phi_0\otimes\psi_1} + \boxed{\phi_0\otimes\psi_2}\\
&\quad + \boxed{\phi_1\otimes\psi_0} + \boxed{\phi_1\otimes\psi_1} + \underbrace{\phi_1\otimes\psi_2}_{\in\mathcal{T}^3(V)} \in\mathcal{T}(V)
\end{aligned}$$

($\phi_0\otimes\psi_1$, $\phi_1\otimes\psi_0$ $\in\mathcal{T}^1(V)$; $\phi_0\otimes\psi_2$, $\phi_1\otimes\psi_1$ $\in\mathcal{T}^2(V)$)

What about $\Lambda(V)$? The product $\otimes$ does not generate altenating tensor

Def. For $\phi \in \mathcal{J}^r(V)$ and $\psi \in \mathcal{J}^s(V)$ we set

$\phi \wedge \psi \in \mathcal{J}^{r+s}(V)$ with

$\phi \wedge \psi := \frac{(r+s)!}{r!\,s!} \mathcal{A}(\phi \otimes \psi)$ called EXTERIOR PRODUCT or WEDGE PRODUCT

Lemma. the Wedge product is bilinear and associative.

Corollory: $\Lambda(V)$ with the wedge product is an associative algebra

called EXTERIOR or GRASSMAN ALGEBRA over V

Lemma. If $\phi \in \Lambda^r(V)$ and $\psi \in \Lambda^s(V)$ then $\phi \wedge \psi = (-1)^{rs} \psi \wedge \phi$

Thm. If $\dim V = n$

1) If $r > n$, then $\Lambda^r(V) = 0$

2) If $0 \leq r \leq n$, then $\dim \Lambda^r(V) = \binom{n}{n} := \frac{n!}{r!(n-r)!}$

In particular if $r=n$, $\dim \Lambda^n(V) = 1 \Rightarrow$ unicity of det

3) $\dim \Lambda(V) = 2^n$

(Next time : $M \mapsto \bigcup_{p \in M} \Lambda(T_pM)$)

II.2 About bases

Recall that if $\{E_1, \cdots, E_n\}$ is a basis of V, then $\exists!$ basis $\{\varphi_1, \cdots, \varphi_n\}$ of V^* s.t.

$\varphi_j(E_k) = \delta_{jk} := \begin{cases} 1, & \text{if } j=k \\ 0, & \text{otherwise} \end{cases}$

$\Rightarrow \forall v \in V : v = \sum_{j=1}^n \underbrace{\varphi_j(v)}_{\text{component of } v \text{ on } E_j} E_j$

We call $\{\varphi_1, \cdots, \varphi_n\}$ the DUAL BASIS.

Consider M a smooth manifold, and (U, φ) a local chart.

For any $p \in U$ a basis of $T_p(M)$ is given by the coordinate frame $\{E_{1,p}, \cdots, E_{n,p}\}$ with

$E_{j,p} := \varphi_*^{-1}\left(\frac{\partial}{\partial x_j}\Big|_{\varphi(p)}\right)$

Thus if we consider the dual space $T_p(M)^* \equiv T_p^*(M)$

there exists a dual basis for $\{E_{1,p}, \cdots, E_{n,p}\}$, usually denoted by $\{(dx^j)_p\}_{j=1}^n$

"Justification" for the notation (change of point of view)

Let $f \in C^\infty(p)$ and $X_p \in T_p(M)$. We set $(df)_p(X_p) := X_p f \in \mathbb{R}$ and in particular

$(df)_p(E_{j,p}) = \left[\varphi_*^{-1}\left(\frac{\partial}{\partial x_j}\Big|_{\varphi(p)}\right)\right](f) = \left[\frac{\partial}{\partial x_j}(\underbrace{f \circ \varphi^{-1}}_{\mathbb{R} \leftarrow \mathbb{R}^n})\right](\varphi(p))$

If we choose $f = \varphi^i : V_p \ni V \mapsto \mathbb{R} \Rightarrow$ $= \frac{\partial}{\partial x_j}(x^i)(\varphi(p)) = \delta_{ij}$

Observe that $(df)_p : T_p(M) \mapsto \mathbb{R}$ is linear, and thus an element of $T_p^*(M)$

$\Rightarrow (d\varphi^i)_p$ is an element of the dual basis.