I.4 Vector fields

We consider a map $X: M \mapsto \bigcup_{p \in M} T_p(M)$,
$p \mapsto X_p \in T_p(M)$

How can one impose some smoothness on X?

1^{st} solution: (best) (too abstract)

consider $T(M) = \bigcup_{p \in M} T_p(M)$ with a certain topology making a smooth manifold.

$T(M)$ is called TANGENT BUNDLE. → describe this: exercise for mathematiciens.

Then consider X as a smooth map between smooth manifolds.

2^{nd} solution:

for a coordinate system (U, φ) on M and for $p \in U$,

we consider the basis $\{E_{j,p}\}_{j=1}^{n}$

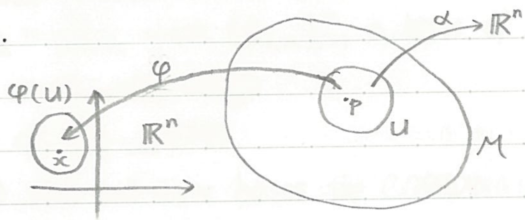
Then $X_p \in T_p(M)$ and $X_p = \sum_{j=1}^{n} \underbrace{\alpha_j(p)}_{\in \mathbb{R}} E_{j,p}$ (a decomposition of X_p on this basis)

By moving p in U, the coefficients $\alpha_j(p)$ is also varying.

So we can impose that

$$\mathbb{R}^n \supset \varphi(U) \ni x \longmapsto (\alpha \circ \varphi^{-1})(x) \in \mathbb{R}^n \text{ is smooth.}$$

This requirement $\Leftrightarrow$ first solution.



Def. a C^∞-VECTOR FIELD on M is a map $X: M \mapsto T(M)$ whose components α_j in the coordinate frame $\{E_{i,p}\}$ of any coordinate system satisfy

$$\mathbb{R}^n \supset \varphi(U) \ni x \longmapsto (\alpha \circ \varphi^{-1})(x) \in \mathbb{R}^n \text{ is smooth.}$$

The set of all C^∞-vector fields is denoted by $\mathfrak{X}(M)$.

Lemma: $X: M \mapsto T(M)$ is a C^∞-vector field iff

$$\forall f \in C^\infty(M, \mathbb{R}): Xf: M \mapsto \mathbb{R}, [Xf](p) \equiv [Xf]_p := X_p f \text{ is smooth.}$$

(another equivalent def) (could be an exercise)

Observe that in this lemma, X can be considered as a map

$$C^\infty(M) \ni f \xrightarrow{X} Xf \in C^\infty(M)$$

Remark: $\mathfrak{X}(M)$ is a vector space and has ~~an~~ additional structures:

1) $\mathfrak{X}(M)$ is a $C^\infty(M)$-MODULE

$:\Leftrightarrow \forall f \in C^\infty(M) \forall X \in \mathfrak{X}(M) : \exists fX \in \mathfrak{X}(M)$ defined by $[fX]_p := f(p) X_p$

2) $\mathfrak{X}(M)$ is a Lie-algebra (very important)

$:\Leftrightarrow$ We can endow $\mathfrak{X}(M)$ with a Lie bracket :

$\mathfrak{X}(M) \times \mathfrak{X}(M) \longmapsto \mathfrak{X}(M)$ given by

$(\underset{X}{\cup\!\!\!\mid}, \underset{Y}{\cup\!\!\!\mid}) \longmapsto [X,Y] := XY - YX$ satisfying

i) linearity in each element

ii) antisymmetry : $[X,Y] = -[Y,X]$

iii) Jacobi identity : $[X,[Y,Z]] = [Y,[Z,X]] = [Z,[X,Y]]$

Exercise : show that 1) and 2) hold

In particular check that $[X,Y]_p$ satisfies Leibniz's rule.

Recall that for any $X_p \in T_p(M)$ $\exists c : (-\varepsilon, \varepsilon) \mapsto M$ with $c(0) = p$ and

$\dot{c}(0) := c_* \left(\frac{d}{dt}\Big|_{t=0}\right) = X_p$

Thm. Let M be a smooth manifold and $X \in \mathfrak{X}(M)$.

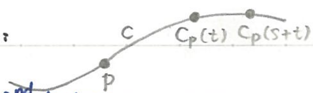
$\forall p \in M \ \underset{\text{there exists a unique}}{\exists!} c_p : (-\varepsilon,\varepsilon) \mapsto M$ with $c_p(0) = p$ and $\dot{c}_p(t) = X_{c_p(t)}$

Remarks: The curve c_p is called the INTEGRAL CURVE of X at p,

and we call $c_p((-\varepsilon,\varepsilon))$ the ORBITAL of p.

⚠ The value ε depends on p.

Whenever it is well-defined, the following relation holds :



$c_p(s+t) = c_{c_p(t)}(s)$

Thm. The orbit of p is either the single point p or an immersion of $(-\varepsilon,\varepsilon)$ in M.

in Appendix for 2nd lecture

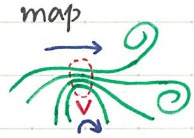
if $X_p = 0$ ... if $X_p \neq 0$

Thm. For any $x \in \mathfrak{X}(M)$ and any $p \in M$ $\exists V \in \mathcal{V}_p$, $\varepsilon > 0$ and a smooth map

$\mathcal{V}_p := \{\text{open sets} \ni p\}$

$F : \underbrace{(-\varepsilon,\varepsilon) \times V}_{\text{s.m.}} \longmapsto M$ satisfying

$F(0,q) = q \in V$ and $\dot{F}(t,q) = X_{F(t,q)}$ $\forall \begin{smallmatrix} t \in (-\varepsilon,\varepsilon) \\ q \in V \end{smallmatrix}$



※ The map F is called the LOCAL FLOW of X at p. Note that $F(t,p) = c_p(t)$

Def. Let $X \in \mathfrak{X}(M)$ and $p \in M$. If $X_p = 0$ then p is called a SINGULAR POINT of the vector field. Since $c_p(t) = p \ \forall t$ if p is sigular, these points are very special and we can study the integral curves around them.

The possible behaviors depend on the topology. Nice subject but we can't go further.

Def. A C^∞-vector field is COMPLETE if

at any $p \in M$, c_p is defined on all $\mathbb{R}$.

A complete vector field can contain some singular points.

Thm. Any C^∞-vector field on a compact manifold is complete.

Remark: Let $X \in \mathfrak{X}(M)$, $p \in M$ and c_p the corresponding integral curve.

Then for any $f \in C^\infty(p)$:

$$X_p f = \frac{d}{dt} f(c_p(t))\Big|_{t=0} = \lim_{t \to 0} \frac{f(c_p(t)) - f(p)}{t}$$

If $f \in C^\infty(M, \mathbb{R})$ recall that

$Xf \equiv L_X f$ is defined by $[Xf]_p = X_p f$

↳ called the LIE DERIVATIVE of f

interpreted as the derivative of f in the direction given by X.

If $Y \in \mathfrak{X}(M)$, the Lie derivative $L_X Y \in \mathfrak{X}(M)$ of Y is defined by

$$[L_X Y]_p := \lim_{t \to 0} \frac{1}{t} \Big(\underbrace{F(-t, \underbrace{V_{c_p(t)}}_{\in V_{c_p(t)}})_*}_{T_p(M) \leftarrow T_{c_p(t)}(M)} \underbrace{Y_{c_p(t)}}_{\in T_{c_p(t)}(M)} - \underbrace{Y_p}_{\in T_p(M)} \Big)$$

$F(-t, \cdot) : \mapsto V_p$

Lemma: $L_X Y = [X, Y]$