I.3 Tangent Space

Recall that a PARAMETRIC SURFACE in $\mathbb{R}^3$ is a map $m: \mathbb{R}^2 \supset \Omega \mapsto \mathbb{R}^3$

Set $M := m(\Omega)$. For $p \in M$ and $c: (-\varepsilon, \varepsilon) \mapsto M \subset \mathbb{R}^3$ with $c(0) = p$ and if c is smooth,

$v := c'(0)$ is TANGENT to M at p.

The set of all such vectors generate the TANGENT PLANE.

Intrinsively, if M is a smooth manifold and if (U, φ) a chart at $p \in M$, then we could set

$v := \left[\frac{d}{dt}(\varphi \circ c)\right](0) \in \mathbb{R}^n$ and call it a tangent vector. (well-defined)

$\mathbb{R}^n \leftarrow M \leftarrow (-\varepsilon, \varepsilon)$

But it depends too much on the choice of a chart.

Def. For $p \in M$ (a s.m.) we denote by $C^\infty(p)$ the EQUIVALENCE CLASS of smooth real functions defined on a neighborhood of p.

Two functions are identified if they coincide (are identically same) on a neighborhood of p.

The elements of $C^\infty(p)$ are called GERMS of C^∞-function at p.

Observations: $C^\infty(p)$ is a vector space with the multiplication of functions

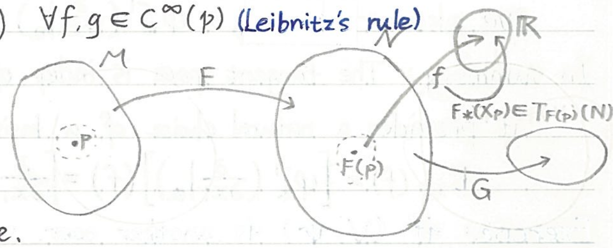
$\Leftrightarrow$ $C^\infty(p)$ is an algebra.

Def. The TANGENT SPACE $T_p(M)$ of M at p is the set of all maps

$X_p : C^\infty(p) \mapsto \mathbb{R}$ satisfying

1) $X_p(\alpha f + g) = \alpha X_p(f) + X_p(g) \quad \forall f, g \in C^\infty(p), \forall \alpha \in \mathbb{R}$

2) $X_p(fg) = X_p(f) \cdot g(p) + f(p) \cdot X_p(g) \quad \forall f, g \in C^\infty(p)$ (Leibnitz's rule)



$T_p(M)$ is endowed with

1) $(X_p + Y_p)(f) := X_p(f) + Y_p(f)$

2) $(\alpha X_p)(f) = \alpha X_p(f)$

which makes $T_p(M)$ a real vector space.

⚠ A tangent vector at p is any $X_p : C^\infty(p) \mapsto \mathbb{R}$.

Observe that this def is indep of any chart, and is intrusic.

Thm. 

(proof as exercise) (simple)

Let $F: M \mapsto N$ be a smooth map between smooth manifolds. For any $p \in M$:

$F^* : C^\infty(F(p)) \mapsto C^\infty(p), \; F^*(f) := f \circ F$

$F_* : T_p(M) \mapsto T_{F(p)}(N), \; [F_*(X_p)](f) := X_p(F^*(f)) = X_p(f \circ F) \quad \forall f \in C^\infty(F(p))$

Then F^* is a homomorphism of algebra $(F^*(f + \alpha g) = F^*(f) + \alpha F^*(g), \; F^*(fg) = F^*(f) F^*(g))$

and F_* is a homomorphism (means preserving structures) of vector space. $(F_*(X_p + \alpha Y_p) = F_*(X_p) + \alpha F_*(Y_p))$

If $H = G \circ F$, $H^* = F^* \circ G^*$ and $H_* = G_* \circ F_*$.

F_* is called the DIFFRENTIAL of F and also denoted by $dF \equiv DF \equiv F'$

Now consider a local version of this result, with $N = \mathbb{R}^n$.

Let $p \in M$ and (U, φ) a coordinate system ($\equiv$ a chart) at p

Then $\varphi_* : T_p(M) \mapsto T_{\varphi(p)}(\mathbb{R}^n)$ is a homomorphism $\forall p \in U$

If $a := \varphi(p) \in \varphi(U)$, then $\varphi^{-1}{}_* : T_a(\mathbb{R}^n) \mapsto T_p(M)$ is a homomorphism

It implies that φ_* and $\varphi^{-1}{}_*$ are isomorphisms.

$\rightsquigarrow$ We can borrow information from $T_a(\mathbb{R}^n)$

$\nearrow : C^\infty(a) \mapsto \mathbb{R}$ $\quad \nearrow$ There exists one unique

Lemma: $\forall X_a \in T_a(\mathbb{R}^n)\ \exists!\ v \in \mathbb{R}^n$ s.t.

$$X_a(f) = \sum_{j=1}^n v_j \left(\frac{\partial F}{\partial x_j}\right)(a) = v \cdot [\nabla f](a) = [D_v f](a) \quad \text{(directional derivative)}$$

and any $v \in \mathbb{R}^n$ defines an element of $T_a(\mathbb{R}^n)$ by $X_a = D_v$.

In other words $T_a(\mathbb{R}^n) \ni X_a \xleftrightarrow{\text{bijective}} v \in \mathbb{R}^n$

$$\underset{\xleftarrow[\text{simple}]{}}{\xrightarrow{\text{less simple}}} \quad \text{(to prove)}$$

We conclude that $T_a(\mathbb{R}^n)$ is of dim n.

A basis of $T_a(\mathbb{R}^n)$ is given by $\left\{ \frac{\partial}{\partial x_1}\Big|_a, \frac{\partial}{\partial x_2}\Big|_a, \cdots, \frac{\partial}{\partial x_n}\Big|_a \right\}$

which can be written by $E_{i,a} = \frac{\partial}{\partial x_i}\Big|_a$ with $\{E_{i,a}\}_{i=1}^n$ a basis of $T_a(\mathbb{R}^n)$

$\Rightarrow$ For any coordinate system (U, φ) on M, the image

$\left\{ \varphi_*^{-1}\left(\frac{\partial}{\partial x_i}\Big|_a\right) \right\}$ is a basis of $T_{\varphi^{-1}(a)}(M)$.

We also write $E_{i,p} = \varphi_*^{-1}\left(\frac{\partial}{\partial x_i}\Big|_a\right)$ and call these bases the COORDINATE FRAMES.

In summary: The tangent space is indep of any coordinate systems, but once one is given it provides a natural choice of a basis, namely if $f \in C^\infty(p)$, then

$$E_{i,p}(f) = \left[\varphi_*^{-1}\left(\frac{\partial}{\partial x_i}\Big|_a\right)\right](f) = \left[\frac{\partial}{\partial x_i}(f \circ \varphi^{-1})\right](\varphi(p))$$

Exercise: if (V, ψ) is another coor. system, what are the relations between these bases?

Corollory: If $F : M \mapsto N$ is smooth and if $p \in M$,

the rank of F at p is equal to the dim of $F_*(T_p(M))$ in $T_{F(p)}(N)$

(another def of rank indep of the coor. systems)

Back to curves:

are smooth manifolds

Consider $c:(-\varepsilon,\varepsilon)\mapsto M$ a smooth map.

On $(-\varepsilon,\varepsilon)$ all tangent vector at $t_0\in(-\varepsilon,\varepsilon)$ are given by $v\,\overbrace{\frac{d}{dt}}^{\text{unit vector}}\Big|_{t_0}$ for $v\in\mathbb{R}$

Then $c_*\left(\frac{d}{dt}\Big|_{t_0}\right)f = \left[\frac{d}{dt}(f\circ c)\right](t_0) =: \circledast$ $\quad (f\in C^\infty(c(t_0)))$

If (U,φ) is a coor. system at $c(t_0)$

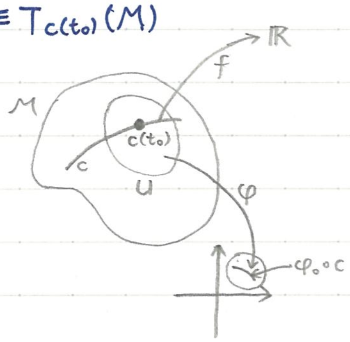
and if we set $c^i := (\varphi\circ c)^i \quad \forall i=1,\cdots,n$

$$\circledast = \left[\frac{d}{dt}\left(\underbrace{f\circ\varphi^{-1}}\circ\underbrace{\varphi\circ c}\right)\right](t_0) = \left[\frac{d}{dt}\left(f\circ\varphi^{-1}(c^1,c^2,\cdots,c^n)\right)\right](t_0) \leftarrow \text{Calculus II}$$

$\mathbb{R}\leftarrow\mathbb{R}^n\leftarrow(-\varepsilon,\varepsilon)$

$$= \sum_{j=1}^{n}\frac{\partial f\circ\varphi^{-1}}{\partial x_j}(\varphi\circ c(t_0))\,c^{j\prime}(t_0) = \sum_{j=1}^{n} c^{j\prime}(t_0)\,E_{j,c(t_0)}(f) \in T_{c(t_0)}(M)$$

$\Rightarrow$ A curve defines an element of $T_{c(t_0)}(M)$



The converse:

<u>Lemma</u> For any $p\in M$ and any $X_p\in T_p(M)$

$\exists c:(-\varepsilon,\varepsilon)\mapsto M$, smooth and with $c(0)=p$, s.t.

$c_*\left(\frac{d}{dt}\Big|_{t=0}\right) = X_p$