In the example on P_2, $B=\{B(x,\frac{1}{m}) \mid x\in\mathbb{Q}^n, m\in\mathbb{N}\}$

Let us define a half-space :

$$H^n := \{(x_1,\cdots,x_n)\in\mathbb{R}^n \mid x_n \geq 0\}$$

$$\partial H^n := \{(x_1,\cdots,x_n)\in\mathbb{R}^n \mid x_n = 0\}$$ for the boundary.

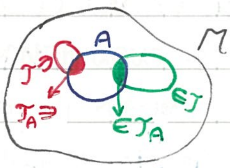
Def. a TOPOLOGICAL MANIFOLD of dimension n with a boundary is a Hausdoff second-countable topological space M with each point $p\in M$ having a neighborhood V eigther homomorphic to an open subset of $H^n\setminus\partial H^n$ or to an open subset of H^n with the image of p inside ∂H^n.

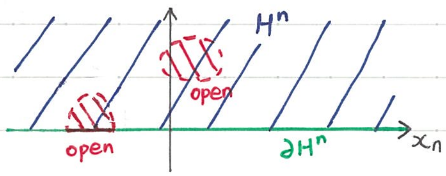
$\boxed{\text{Remark}}$: If $(M,\mathcal{T})$ is a topo. space and if $A\subset M$.

Then the topology on A is given by $\mathcal{T}_A := \{V\cap A \mid V\in\mathcal{T}\}$

(called RELATIVE or SUBSPACE TOPOLOGY)

⚠ An open set for A (in $\mathcal{T}_A$) is not always an open set for M (in $\mathcal{T}$).





$\swarrow C^{\infty}$

I.2 Smooth manifolds & Smooth maps

Def. a SMOOTH (or C^{∞}) MANIFOLD M is a topo. manifold together with a family of homeomorphisms $\varphi_\alpha : \mathbb{R}^n \supset \underset{\text{open}}{U_\alpha} \longmapsto M$ s.t.

1) $\bigcup_\alpha \varphi_\alpha(U_\alpha) = M$
2) If $\varphi_\alpha(U_\alpha) \cap \varphi_\beta(U_\beta) =: V_{\alpha\beta} \neq \phi$ then

$$\left\{\begin{array}{l} \varphi_\beta^{-1}\circ\varphi_\alpha : \varphi_\alpha^{-1}(V_{\alpha\beta}) \mapsto \varphi_\beta^{-1}(V_{\alpha\beta}) \\ \varphi_\alpha^{-1}\circ\varphi_\beta : \varphi_\beta^{-1}(V_{\alpha\beta}) \mapsto \varphi_\alpha^{-1}(V_{\alpha\beta}) \end{array}\right. \quad \text{(TRANSITION FUNCTIONS)}$$

are of C^{∞} (from a subset of $\mathbb{R}^n$ to a subset of $\mathbb{R}^n$).

3) The family $\mathcal{A} = \{(U_\alpha, \varphi_\alpha)\}_\alpha$ is maximal.

$\mathcal{A}$ is called a C^{∞} $\underbrace{\text{MAXIMAL}}_{3)}$ $\underbrace{\text{ATLAS}}_{1)\,\&\,2)}$.

MAXIMAL: If $\varphi : \overset{\subset\mathbb{R}^n,\ \text{open}}{U} \mapsto M$ satisfies $\varphi^{-1}\circ\varphi_\alpha$ and $\varphi_\alpha^{-1}\circ\varphi$ (whenever defined) is smooth then $(U, \varphi) \in \mathcal{A}$.

Remark: ◦ it is often easy to describe an atlas, but not the maximal one.

◦ A topological manifold can be endowed with different inequivalent maximal atlases. (see the P_1 on today's handout) (very deep)

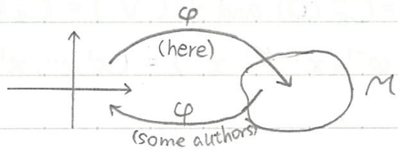
INEQUIVALENT: take 2 max atlases, if the union is not an atlas (some transision functions are not C^{∞}) then the 2 atlases are not equivalent.

Exercises

1) Provide an example of smooth manifolds with an atlas.
   (n-sphere, group of matrices, Lie groups, real projective space $P(\mathbb{R}^n)$, etc)
2) Show the uniqueness (starting from a certain atlas) of the maximal atlas.
3) Look at inequivalent atlases on n-sphere.
   $\parallel$
   differential structure

Remark:

1)



2) For $(U, \varphi) \in \mathcal{A}$ and $p \in \varphi(U) \subset M$, we set

$$\varphi^{-1}(p) =: (x^1(p), x^2(p), \cdots, x^n(p))$$

and call it a LOCAL COORDINATE of p. It means

$\varphi^{-1}(\cdot) =: (x^1(\cdot), x^2(\cdot), \cdots, x^n(\cdot))$ (a CHART or a LOCAL COORDINATE FUNCTION)

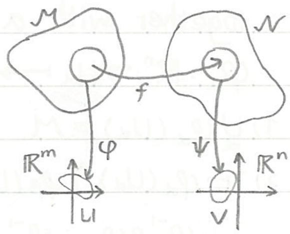
is an homeomorphism from an open subset of M to an open subset of $\mathbb{R}^n$.

Def: Let $\mathcal{M}, \mathcal{N}$ be smooth manifolds of dim m and n respectively.

A map $f: \mathcal{M} \mapsto \mathcal{N}$ is a SMOOTH MAP if

$\forall$ charts (U, φ) of $\mathcal{M}$ and (V, ψ) of $\mathcal{N}$:

$\psi \circ f \circ \varphi^{-1}$ is smooth wherever defined.



The function $\psi \circ f \circ \varphi^{-1}$ is called a LOCAL REPRESENTATION of f.

We set $C^\infty(\mathcal{M}, \mathcal{N}) :=$ the set of such smooth functions,

and $C^\infty(\mathcal{M}) := C^\infty(\mathcal{M}, \mathbb{R})$.

Def. If $f \in C^\infty(\mathcal{M}, \mathcal{N})$ is bijective and if $f^{-1} \in C^\infty(\mathcal{N}, \mathcal{M})$, we call f a DIFFEOMORPHISM.

Remark: • a diffeomorphism is also a homeomorphism.

• A map $f: \mathcal{M} \mapsto \mathcal{N}$ is a LOCAL DIFFEOMORPHISM at $p \in \mathcal{M}$ if

$\exists V \in \mathcal{V}_p$ and $W \in \mathcal{V}_{f(p)}$: $f|_V : V \mapsto W$ is a diffeomorphism.

Def. Let $f: \mathcal{M} \mapsto \mathcal{N}$ be a smooth function and let (U, φ) (V, ψ) be charts of $\mathcal{M}$ & $\mathcal{N}$ respectively.

For $p \in \mathcal{M}$, the RANK of f at p $(=: \mathrm{rank}(f)_p)$ corresponds to

the rank of the Jacobian matrix

$$\begin{pmatrix} \frac{\partial F_1}{\partial x_1} & \cdots & \frac{\partial F_1}{\partial x_m} \\ \vdots & & \vdots \\ \frac{\partial F_n}{\partial x_1} & \cdots & \frac{\partial F_n}{\partial x_m} \end{pmatrix} (\varphi(p)) \quad \text{with } F := \psi \circ f \circ \varphi^{-1}$$

This rank is independent of the charts.

Thm. (not so easy) Framework as before. (Constant rank thm)

Suppose that $\mathrm{rank}(f)_p = k$ $\forall p \in \mathcal{M}$, with $k \in \mathbb{N}$. Then

$\forall p \in \mathcal{M} \; \exists (U, \varphi), (V, \psi)$ charts of $\mathcal{M}, \mathcal{N}$ respectively s.t.

◦ $\varphi(p) = \mathbf{0} \in \mathbb{R}^m$ and $\psi(f(p)) = \mathbf{0} \in \mathbb{R}^n$;

◦ $\varphi(U) = C^m_\varepsilon(\mathbf{0})$ and $\psi(V) = C^n_\varepsilon(\mathbf{0})$ $\exists \varepsilon > 0$; Cube in $\mathbb{R}^n$ centered at $\mathbf{0}$ and with $x^i \in (-\varepsilon, \varepsilon)$

◦ $\psi \circ f \circ \varphi^{-1}(x^1, \cdots, x^m) = (x^1, \cdots, x^k, \underbrace{0, \cdots, 0}_{n-k})$