VI General relativity

Def. a PSEUDO-RIEMANNIAN MANIFOLD is a pair (M, ϕ) with
M a smooth manifold and $\phi \in \mathcal{T}^2(M)$, symmetric and non-degenerate.

$\phi(X,Y) = \phi(Y,X)$ ⁝ $\phi(X,Y) = 0 \ \forall Y \in \mathfrak{X}(M) \Downarrow X = 0$

⚠ No 「positive definite」 required !

a LORENTZIAN MANIFOLD is a pseudo-Riemannian manifold

with $(g_{ij}) = \begin{pmatrix} -1 & & 0 \\ & 1 & \\ 0 & & \ddots \ 1 \end{pmatrix}$ in suitable coordinates (locally). → signature (trace) $= n-2$

Facts for pseudo-Riemannian manifolds

1) unicity of Levi Civita connection when the 2 conditions are imposed.
2) Koszul formula still holds.
3) Hopf-Rinow thm + geodesically complete are no more valid.
   ⇒ We don't have a metric space anymore.
4) Cartan structure thm are still valid.

Recall that the length of a vector is not changed under parallel transport along a curve.

Geodesics c satisfy that $\dot{c}$ is parallel transported along c.

$\Rightarrow \phi(\dot{c}, \dot{c}) = cst$

Def. A geodesic c on a pseudo-Riemannian manifold (M, ϕ) is
TIMELIKE, NULL, or SPACELIKE if
$\phi(\dot{c},\dot{c}) < 0$, $\phi(\dot{c},\dot{c}) = 0$ or $\phi(\dot{c},\dot{c}) > 0$

<0 and $=0$ are allowed by PSEUDO METRIC ϕ

Remark: these expressions come from special relativity with $M = \mathbb{R}^4$ and

$$\underset{\mu,\nu = 0,1,2,3}{(g_{\nu\mu})} = \begin{pmatrix} -1 & & 0 \\ & 1 & \\ 0 & & 1 \ \ 1 \end{pmatrix}$$

a special case of a Lorentzian manifold.

For a Lorentzian manifold (M, ϕ) of dim 4, with the Levi Civita connection, the Einstein field equation reads

$$\underbrace{\underset{\substack{\uparrow \\ \text{Ricci curvature}}}{R_{\mu\nu}} - \tfrac{1}{2}\underset{\substack{\uparrow \\ \text{scalar curvature}}}{R}g_{\mu\nu}}_{\substack{\text{(about geometry)}\ G_{\mu\nu} \text{ Einstein tensor}}} + \underset{\substack{\uparrow \\ \text{cosmological constant}>0}}{\Lambda} g_{\mu\nu} = \frac{8\pi G}{c^4} \underset{\substack{\uparrow \\ \text{stress-energy or energy-momentum tensor}}}{T_{\mu\nu}} \quad \text{for } \mu,\nu = 0,1,2,3 \qquad ⊛$$

contains the physics (energy + matter)

G = gravitation constant
c = speed of light

⚠ Not so much freedom for writing a meaningful equations.

This is a system of 10 equations because of symmetry between μ and ν.

In addition the thms of structure reduces the number of indep eq.

Remark: These equations define the pseudo metric tensor $g_{\mu\nu}$.
Indeed, $R_{\mu\nu\gamma}{}^{\delta}$ and $R_{\mu\nu}$ can be expressed in terms of $\Gamma_{\mu\nu}{}^{\delta}$ and its derivativ
And $\Gamma_{\mu\nu}{}^{\delta}$ can be expressed in terms of $g_{\mu\nu}$ and its derivatives.

$\Rightarrow$ ⊛ is a system of non linear partial differential equations for $g_{\mu\nu}$.

Schwarzschild solution

Assumptions:
- $T_{\mu\nu} = 0$
- $g_{\mu\nu}$ is time independent (static solution)
- spherically symmetric in space ($\equiv$ in the indices 1,2,3)
- $M = \mathbb{R} \times \mathbb{R}_+ \times \mathbb{S}^2$. $\mathbb{R}$ for t and $\mathbb{R}_+ \times \mathbb{S}^2$ is $\mathbb{R}^3$ in spherical coordinates

Suppose that

$$g = -A^2(r)\, dt \otimes dt + B^2(r)\, dr \otimes dr + r^2 d\theta \otimes d\theta + r^2 \sin^2(\theta)\, d\varphi \otimes d\varphi \in \mathcal{J}^2(M)$$

with $A, B : \mathbb{R}_+ \longmapsto \mathbb{R}$ unknown,
$(dt, dr, d\theta, d\varphi) \in \mathcal{J}^1(M)$ generate a basis of $T^*(M)$.
$\{(\frac{\partial}{\partial x^i})_p\}_{i=1}^n$ is a basis of $T_p(M)$, and $\{dx_p^i\}_{i=1}^n$ is a basis of $T_p^*(M)$.
$\Rightarrow \{dx^i \otimes dx^j\}_{i,j}$ is a basis of $\mathcal{J}^2(M)$.

Question: can we find A, B such that ⊛ is satisfied (with $T_{\mu\nu} = 0$)?

Two approaches:

1) Express $\Gamma_{\mu\nu}{}^{\delta} \rightsquigarrow R_{\mu\nu\delta}{}^{\gamma}$ and then $R_{\mu\nu}$ and R in terms of $g_{\mu\nu}$, and solve ⊛

2) Set
$$\left.\begin{matrix} \theta^0 := A(r)\, dt & \theta^2 := r\, d\theta \\ \theta^1 := B(r)\, dr & \theta^3 := r\sin(\theta)\, d\varphi \end{matrix}\right\} \in \mathcal{J}^1(M)$$
and observe that
$g = \sum_{\mu,\nu} \eta_{\mu,\nu}\, \theta^{\mu} \otimes \theta^{\nu}$ and $\eta_{\mu\nu} = \begin{pmatrix} -1 & & & 0 \\ & 1 & & \\ & & 1 & \\ 0 & & & 1 \end{pmatrix}$ and that
$\{\theta^0, \theta^1, \theta^2, \theta^3\}$ is an orthonormal coframe a basis of $T^*(M)$

- Define $\theta_{\mu}{}^{\nu}$ and $\Omega_{\mu}{}^{\nu}$ (connection and curvature tensors) and write the structure relations of Cartan.
- One obtains some differential equations for A and B, which can be solved.
- $A(r) = \left(1 - \frac{2m}{r}\right)^{\frac{1}{2}}$ and $B(r) = \left(1 - \frac{2m}{r}\right)^{-\frac{1}{2}}$ with $m \in \mathbb{R}$ an integration const

Conclusion

Textbooks on general relativity are now accessible
(but still the theory is complicated).