V.2 Equation of structure

Recall that a connection ∇ is a map

$$\nabla: \mathfrak{X}(M) \times \mathfrak{X}(M) \longmapsto \mathfrak{X}(M)$$
$$X \quad Y \longmapsto \nabla_X Y$$

which is bilinear and satisfies

1) $\nabla_{fX} Y = f \nabla_X Y$
2) $\nabla_X (fY) = (Xf) Y + f \nabla_X Y$

∇ is torsion free if $\nabla_X Y - \nabla_Y X - [X,Y] \; (=: T(X,Y)) = 0$ and

∇ is compatible with the metric $\overset{\text{for}}{(M,\phi)}$ if

$Z\langle X,Y\rangle = \langle \nabla_Z X, Y\rangle + \langle X, \nabla_Z Y\rangle$ making the parallel transport of 2 orthogonal vectors still orthogonal $\Bigg\} \circledast$

Let U be an open subset of M and

let $\{F_j\}_{j=1}^n$ be a C^∞-field of frames on U $\{F_{j,p}\}$ is a basis of $T_p(M)$ $\forall p \in U$; not necessarily orthonormal nor generated by a chart

e.g. the coordinate frames given by a chart (U, φ)

Let $\{\theta^j\}_{j=1}^n$ be a dual coframe, it means $\{\theta^j\}$ is a C^∞-field of frames on $T(M)^*$

and $\{\theta^j_p\}$ is a basis of $T_p(M)^*$ with $\theta^j_p(F_{k,p}) = \delta_{jk}$ Cronecker delta

Recall that ∇ is uniquely determined by $\{\Gamma_{ij}{}^k\}$ defined by

$$\nabla_{F_i} F_j = \sum_k \Gamma_{ij}{}^k F_k$$

Def. $\theta_j{}^k := \sum_l \Gamma_{lj}{}^k \theta^l \in \mathcal{J}^1(M)$ one form

$\{\theta_j{}^k\}$ are called CONNECTION FORMS. Clearly

$\Rightarrow$ $\theta_j{}^k(F_i) = \Gamma_{ij}{}^k$, and

if $T(M) \ni X = \sum_l b^l F_l$ then

$$\nabla_X F_j = \nabla_{\sum_l b^l F_l} F_j \overset{\text{linear and 1)}}{=} \sum_l \overset{\in C^\infty(U)}{b^l} \nabla_{F_l} F_j \overset{\Gamma}{=} \sum_l b^l \sum_k \Gamma_{lj}{}^k F_k$$
$$\overset{\text{Def of }\theta}{=} \sum_k \sum_l b^l \theta_j{}^k(F_l) F_k \overset{\text{linearity of 1 forms}}{=} \sum_k \theta_j{}^k(X) F_k$$

Thus, $\theta_j{}^k(X)$ are the components of $\nabla_X F_j$ with respect to $\{F_k\}$.

For a R_0 manifold (M,ϕ) and for the Levi Civita connection ∇,

the n^2 connection form are not indep because of the relations $\circledast$.

Thm. (Structure Thm of Cartan) [GN p. 133]

Let (R, ϕ) be a R_o manifold, ∇ the Levi Civita connection, $U, \{E_i\}, \{\theta^j\}$ above.

Then the connection forms $\{\theta_j{}^k\}$ are the unique solution of the equations:

1) $d\theta^i = \sum_j \theta^j \wedge \theta^i$ wedge product — equality between 2-forms

2) $\underbrace{dg_{ij}}_{\in C^\infty(M),\ \text{one-forms}} = \sum_k (g_{kj}\theta_i{}^k + g_{ki}\theta_j{}^k)$ equality between 1-forms

Remark: If $\{F_j\}$ is an orthonormal basis,

$g_{ij} := \phi(F_i, F_j) = \delta_{ij}$ and 2) becomes

2) $0 = \theta_i{}^j + \theta_j{}^i$

Similarly, one can introduce the CURVATURE FORM for $X, Y \in \mathfrak{X}(M)$:

$$\Omega_k^l(X,Y) := \underset{\in \mathfrak{X}(M)^*}{\theta^l}(\underbrace{R(X,Y)F_k}_{\in \mathfrak{X}(M)}) \in C^\infty(M) \Rightarrow \Omega_k^l \in \mathcal{J}^2(U) \subset \mathcal{J}^2(M)$$

which gives

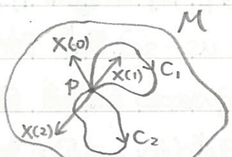
$$R(X,Y)F_k = \sum_j \Omega_k^j(X,Y)F_j$$

Thus $\Omega_k^l(X,Y)$ are components of $R(X,Y)F_k$ on the basis $\{F_j\}$

Remark: $\Omega_k^l \in \mathcal{J}^2(U) \subset \mathcal{J}^2(M)$ and one has

Thm. (Structure Thm of Cartan) [GN p. 135 ; Bo p. 391]

$$\Omega_i{}^j = d\theta_i{}^j - \sum_k \theta_i{}^k \wedge \theta_j{}^k$$ equality between 2-forms



V.3 Holonomy for a connected Riemannian manifold

⚠ Exists in a more general context of vector bundles or principal bundles.

Let $c : [0,1] \ni t \longmapsto c(t) \in M$ a smooth curve on (M, ϕ)

with $c(0) = c(1) = p$.

Let $X_p \in T_p(M)$ and let $X(t)$ be the parallel transport of X_p along c

with $X(0) = X_p$. Let

$P_c : T_p(M) \ni X_p = X_0 \longmapsto X_1 \in T_p(M)$, and clearly

$\underset{\text{composition of paths}}{P_{c_2 \circ c_1}} = P_{c_2} P_{c_1}$; $\underset{\text{backward}}{P_{c^{-1}}} = P_c^{-1}$ leading to the fact that it composes a group.

In addition $P_c \in GL(T_p(M))$ (invertible matrices) because the parallel transport is a solution of a homogenous equation. $\Rightarrow$ linear in the initial condition

In fact $P_c \in O(T_p(M))$ orthogonal matrices on $T_p(M)$.

because the parallel transport preserves norms and scalar products.

Remark: Instead of smooth curve, we can consider C^1-piecewise curves.

We have obtained that

$\{P_c\}_* \subset O(T_p(M))$ is a group e is the zero path

called the HOLONOMY GROUP at p and denoted $\mathrm{Hol}(p)$.

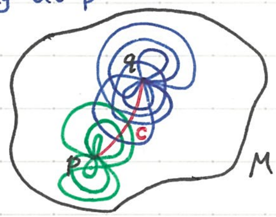
$* :=$ "c any C'-piecewise curve starting and ending at p"

If p and q are 2 points on M then

$\mathrm{Hol}(p)$ is isomorphic to $\mathrm{Hol}(q)$ since

$\mathrm{Hol}(p) = P_c^{-1}\, \mathrm{Hol}(q)\, P_c$

for some fixed path c between q and p.



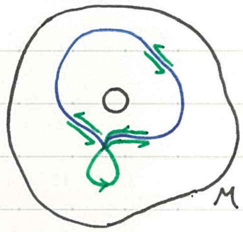
Def. We set $\mathrm{Hol}(M) = \mathrm{Hol}(p) \subset O(\ \ )$ for a fixed $p \in M$, and call it the HOLONOMY GROUP of M.

We also set $\mathrm{Hol}^\circ(M)$ constructed only with C'-piecewise path which can be deformed to the zero path.

Remarks:

1) These groups are representations of the group of paths on M.

2) $\mathrm{Hol}^\circ(M)$ is a normal subgroup of $\mathrm{Hol}(M)$.



Lemma

M is orientable iff $\mathrm{Hol}(M) \subset SO(M)$ determinant 1 and never −1

Thm (deep notation)

$\mathrm{Hol}^\circ(M)$ is compact (it is a closed set in $O(n)$)

Remark [see App. 12]

There is a link between holonomy and the curvature tensor $R(X,Y)$